*ANEKANT EDUCATION SOCIETY'S*

# *TULJARAM CHATURCHAND COLLEGE*

***OF ARTS, SCIENCE AND COMMERCE, BARAMATI***

## *AUTONOMOUS*

# QUESTION BANK

*FOR*

## *M.Sc. Part- I (Sem-I)*

# STATISTICS

## STAT4101: Mathematical Analysis

(With effect from June 2019)

**Q.1.** Choose the correct alternative: **(1 each)**

i) An open cover is a collection of

a) all open sets  b) all closed sets  c) all compact sets  d) all finite sets

ii) Which of the following statement is wrong?

a) Closure of E is closed always  b) Every point of closed set is limit point

c) Complement of open set is closed  d) finite intersection of open set is open

iii) Let z and w be any two complex numbers. Then which of the following not true?

a) $|zw| = |z||w|$  b) $\overline{z+w} = \bar{z} + \bar{w}$  c) $|z| + |w| \le |z+w|$  d) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

iv) The series I) $\sum \frac{1}{2+3n^2}$  II) $\sum \frac{1+n}{2+3n^2}$

a) Only series I is convergent  b) Both series are convergent

c) Only series II is convergent  d) Both series are divergent

v) Which of the following function is a metric?

a) $d(x,y) = \min\{|x|,|y|\} \quad \forall x,y \in \mathbb{R}$  b) $d(x,y) = |x^2 - y^2|, \quad \forall x,y \in \mathbb{R}$

c) $d(x,y) = |x - 2y| \quad \forall x,y \in \mathbb{R}$  d) $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|, \forall x,y \in \mathbb{R} - \{0\}$

vi) Consider a sequence $\{x_n\}_{n=1}^{\infty} = \left\{(-a)^n\right\}$ then lim sup $x_n$ and lim inf $x_n$ is............. respectively.

a) -a and a  b) -a and -a  c) a and a  d) a and -a

vii) If $S_1 = \sqrt{2}, \text{and } S_{n+1} = \sqrt{2 + S_n}, n = 1, 2, .....$ then $\{S_n\}$ converges to

a) 2  b) $2 + \sqrt{2}$  c) $\sqrt{2+\sqrt{2}}$  d) $\sqrt{2}$

viii) Consider the set $E = \bigcap_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{1}{n}\right)$ then

a) E is closed and bounded  b) E is neither open nor closed

c) E is open  d) E is not closed

ix) The interval of convergence of the series $\sum_{n=1}^{\infty} \frac{n^2 x^n}{5^n \sqrt{n^5}}$ is :

a) (-5, 5]  b) [-5, 5)  c) [-5, 5]  d) (-5, 5)

x) Let $\{S_n\}$ be a sequence of real numbers. Then which of the following is true always?

a) $\inf\limits_{n\geq 1}\sup\limits_{k\geq n} s_k \leq \sup\limits_{n\geq 1}\inf\limits_{k\geq n} s_k$  b) $\inf\limits_{n\geq 1}\sup\limits_{k\geq n} s_k \geq \sup\limits_{n\geq 1}\inf\limits_{k\geq n} s_k$

c) $\inf\limits_{n\geq k}\sup\limits_{k\geq 1} s_k \leq \sup\limits_{n\geq k}\inf\limits_{k\geq 1} s_k$  d) $\inf\limits_{n\geq k}\sup\limits_{k\geq 1} s_k \geq \sup\limits_{n\geq k}\inf\limits_{k\geq 1} s_k$

xi) Which of the following function is a metric?

a) $d(x,y)=\max\{|x|,|y|\} \quad \forall x,y\in\mathbb{R}$  b) $d(x,y)=|x^2-y^2|, \quad \forall x,y\in\mathbb{R}$

c) $d(x,y)=|x-y| \quad \forall x,y\in\mathbb{R}$  d) $d(x,y)=|x-2y|, \forall x,y\in\mathbb{R}$

xii) An arbitrary intersection of closed set

a) is open set  b) need not be closed set

c) is closed set  d) need not be open set

xiii) If the series I) $\sum\frac{(-1)^n}{n}$ and series II) $\sum\frac{1}{n}$ then

a) Series I is convergent and series II is divergent.

b) Series I is divergent and series II is convergent.

c) Both series are convergent.

d) Both series are divergent.

xiv) If set E is closed and bounded in Real set then set E is

a) perfect set  b) compact set  c) open set  d) countable set

xv) Consider a sequence $\{x_n\}_{n=1}^{\infty}=\{(-1)^n\}$ then lim inf $x_n$ and lim sup $x_n$ is ............. respectively.

a) -1 and 1  b) -1 and -1  c) 1 and 1  d) 1 and -1

xvi) An arbitrary intersection of closed set

a) is open set  b) need not be closed set

c) is closed set  d) need not be open set

xvii) Which of the following set is closed but not compact?

a) $\{\frac{1}{n} : n\in\mathbb{N}\}$  b) $\{\frac{1}{n} : n\in\mathbb{N}\}$  c) $[0,\infty)$  d) $\{0\}$

xviii) If sequence $s_n=\frac{(-1)^n}{n}$ then lim inf $s_n$ and lim sup $s_n$ are ………. respectively.

a) 0, ½  b) -1, ½  c) -1, 0  d) 0, 0

xix) Which of the following is a pair of open covering and finite sub covering for a set [0, 1].

a) $\{\left(\frac{-1}{n},1+\frac{1}{n}\right)\}_{n=1}^{\infty}$ *and* $\{\left(\frac{-1}{n},1+\frac{1}{n}\right)\}_{n=1}^{5}$  b) $\{\left(\frac{-1}{n},1-\frac{1}{n}\right)\}_{n=1}^{\infty}$ *and* $\{\left(\frac{-1}{n},1-\frac{1}{n}\right)\}_{n=1}^{5}$

c) $\{\left(\frac{1}{n},1-\frac{1}{n}\right)\}_{n=1}^{\infty}$ *and* $\{\left(\frac{1}{n},1-\frac{1}{n}\right)\}_{n=1}^{5}$  d) $\{\left(\frac{1}{n},1+\frac{1}{n}\right)\}_{n=1}^{\infty}$ *and* $\{\left(\frac{1}{n},1+\frac{1}{n}\right)\}_{n=1}^{5}$

xx) $\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n$ equals

a) 1 b) $e^{-2}$ c) $e^{-1/2}$ d) $e^{-1}$

xxi) Let $f : R \to R$ be a monotone function. Then

a) $f$ has no discontinuities

b) $f$ has only finitely many discontinuities

c) $f$ can have at most countably many discontinuities

d) $f$ can have uncountably many discontinuities

xxii) $\lim_{n\to\infty}\log\left(1-\frac{1}{n}\right)^n$ equals

a) 1 b) 0 c) e d) -1

xxiii) If $\sum a_n$ is convergent but not absolutely convergent and $\sum a_n = 0$. Let $s_n$ be partial sum $\sum_{i=1}^{n} a_i$ , n= 1, 2, 3, ... Then

a) $S_n = 0$ for infinitely many n b) $s_n > 0$ for finitely many n

c) $s_n > 0$ for infinitely many n d) $s_n > 0$ for all $n \geq 1$

**Q.2.** State whether the following statements are **TRUE** or **FALSE** with justification. **(1 each)**

i) If sequence $\{x_n, n \geq 1\}$ is bounded above and monotonic increasing then sequence $\{x_n\}$ is converges to lim sup $x_n$.

ii) Absolute convergence of series implies convergence of series.

iii) Every bounded sequence is convergent sequence.

iv) Every differentiable function is continues function.

v) Arbitrary intersection of open set need not be open set.

vi) Set of rational number is ordered set but not countable.

vii) Convergence of series implies absolute convergence of series.

viii) Every bounded sequence is convergent.

ix) Limit inferior and limit superior of a sequence are always exists.

x) Every Cauchy sequence is convergent sequence.

xi) Set A is compact set if and only if its complement is closed set.

xii) Set of irrational number is uncountable set.

xiii) Every bounded monotonic sequence is convergent.

xiv) Limit inferior never exceeds limit superior of a sequence.

xv) Every convergent sequence is Cauchy sequence.

xvi) Set A is open if and only if its complement is closed set.

xvii) Countable union of uncountable sets is countable.

xviii) If sequence $\{x_n, n \geq 1\}$ is bounded below and monotonic deceasing then sequence $\{x_n\}$ is converges to lim inf $x_n$.

xix) Interior and exterior points of a set cannot be boundary point.

xx) Absolute convergence of series implies and implied by convergence of series.

xxi) In any metric space every Cauchy sequence is convergent sequence.

xxii) Set A is compact set if and only if its complement is closed set and bounded.

xxiii) If real valued function $f$ is continuous then it must be differentiable.

xxiv) Set $E = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ is Closed set.

xxv) If F = {0, 1}. Then (F, +, ·) is field.

xxvi) Every bounded sequence is convergent.

**Q. 2**. Define the following terms and give an illustration of each. **(Each- 1)**

1. An ordered set
2. Lower bound of a set
3. Upper bound of a set
4. Greatest lower bound of a set
5. Least upper bound of a set
6. A bounded below set
7. A bounded above set
8. Abounded set
9. Field
10. An ordered field
11. Union of two sets
12. Union of finite number of sets
13. Union of Countable number of sets
14. Intersection of two sets
15. Intersection of finite number of sets
16. Intersection of Countable number of sets
17. Function
18. Injective (or one-one) function
19. Surjective (or onto) function
20. Bijective function
21. Inverse function
22. composition of functions
23. Increasing function
24. Decreasing function
25. Finite set
26. Infinite Set
27. Countable set
28. Uncountable set
29. Metric
30. Metric Space
31. Neighborhood of a point $x$
32. Limit point
33. Interior point
34. Boundary point
35. Closed set
36. Open set
37. Compact set
38. Open cover of a set
39. Finite subcover of a set
40. Closure of a set
41. Sequence of real numbers
42. Convergent sequence
43. Subsequence of a sequence
44. Subsequential limits
45. Cauchy sequence
46. Monotonically increasing sequence
47. Monotonically decreasing sequence
48. Monotonic sequence
49. Limit Superior
50. Limit inferior
51. Series
52. Convergent
53. Power series
54. Series
55. Radius of convergent
56. Absolute convergent
57. Rearrangement of series
58. Limit of a function
59. Continuous function
60. Discontinuous function
61. Discontinuity of first kind
62. Discontinuity of second kind
63. Monotonic function
64. Derivative of a real function
65. Local maximum of a function
66. Local minimum of a function
67. Riemann integral
68. Riemann Stieltjes integral
69. Upper and lower Riemann integral
70. Upper and lower Riemann Stieltjes integral
71. Partition of an interval
72. Norm of partition
73. Upper Riemann Sum
74. Lower Riemann Sum

**Q. 3.** Prove the following theorems:

Theorem 1) Let $f : X \to Y$ be a mapping and let $A \subset X$, $B \subset Y$. Then

i) $f(\phi) = \phi$

ii) $A \subset B \Rightarrow f(A) \subset f(B)$

iii) $f(A \cup B) = f(A) \cup f(B)$

iv) $f(A \cap B) \subset f(A) \cap f(B)$ (Converse is not true. Give suitable example where equality not holds) **(1 M each)**

Theorem 2) Let $f : X \to Y$ be a mapping and let $A_1$ and $A_2$ are subsets of Y. Then we have the following proerties: **(1 M each)**

i) $f^{-1}(\phi) = \phi$ and $f^{-1}(Y) = X$

ii) $A_1 \subset A_2 \Rightarrow f^{-1}(A_1) \subset f^{-1}(A_2)$

iii) $f^{-1}(A_1 \cup A_2) = f^{-1}(A_1) \cup f^{-1}(A_2)$

iv) $f^{-1}(A_1 \cap A_2) = f^{-1}(A_1) \cap f^{-1}(A_2)$

v) $f^{-1}(A^c) = (f^{-1}(A))^c$

Theorem 3) Suppose S is an ordered set with the least upper bound property, $B \subset S$, B is nonempty and B is bounded below. Let L be the set of all lower bounds of B. Then $\alpha = \text{Sup}(L)$ exists in S and $\alpha = \inf(B)$ exists in S. **(4 M)**

Theorem 4) Suppose S is an ordered set with the greatest lower bound property, $B \subset S$, B is nonempty and B is bounded below. Let U be the set of all upper bounds of B. Then $\alpha = \inf(U)$ exists in S and $\alpha = \sup(B)$ exists in S. **(4 M)**

Theorem 5) Archimedean Property: If $x \in \mathcal{R}$, $y \in \mathcal{R}$ and $x > 0$, then there is a positive integer n such that $nx > y$. **(2 M)**

Theorem 6) $\mathcal{Q}$ dense in $\mathcal{R}$ : Between any two real numbers there is a rational number. *i.e.* If $x \in \mathcal{R}$, $y \in \mathcal{R}$ and $x < y$, then there is a $p \in \mathcal{Q}$ such that $x < p < y$. **(2 M)**

Theorem 7) For every real $x > 0$ and every integer $n > 0$ there is one and only one real $y$ such that $y^n = x$. (This number $y$ is written as $\sqrt[n]{x}$ *or* $x^{1/2}$ ). **(5 M)**

Theorem 8) If $a$ and $b$ are positive real numbers and n is a positive integer, then $(ab)^{1/n} = a^{1/n} b^{1/n}$ . **(2 M)**

Theorem 9) Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{R}^k$, and $\alpha$ is any real number, then

i) $|\mathbf{x}| \geq 0$;

ii) $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = 0$;

iii) $|\alpha\, \mathbf{x}| = |\alpha|\, |\mathbf{x}|$;

iv) $|\mathbf{x}\,\mathbf{y}| = |\mathbf{x}||\mathbf{y}|$;

v) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;

vi) $|\mathbf{x} - \mathbf{y}| \leq |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}|$. **(1 M each)**

Theorem 10) Every infinite subset of a countable set A is countable. **(4 M)**

Theorem 11) Countable union of countable sets is countable. **(4 M)**

Theorem 12) Let A be a countable set, and let $B_n$ be theset of all n-tuples ($a_1$, $a_2$, . . ., $a_n$), where $a_k \in$ A (k =1, 2, . . ., n) and the elements $a_1$, $a_2$, . . ., $a_n$ need not distict. Then $B_n$ is countable set. **(5 M)**

Theorem 13) The set of rational numbers is countable. **(2 M)**

Theorem 14) Let A be the set of all sequences whose elements are the two digits 0 and 1. This set A is uncountable. **(2 M)**

Theorem 15) Set of real number in [0, 1] is uncountable. **(2 M)**

Theorem 16) Every neighborhood is an open set. **(4 M)**

Theorem 17) If p is a limit point of a set E, then every neighborhood of $p$ contains infinitely many points of E. **(4 M)**

Theorem 18) A set is open if and only if its compliment is open. **(4 M)**

Theorem 19) Arbitrary union of open sets is open. **(2 M)**

Theorem 20) Arbitrary intersection of closed sets is closed. **(2 M)**

Theorem 21) Finite intersection of open sets is open. **(2 M)**

Theorem 22) Finite union of closed sets is closed. **(2 M)**

Theorem 23) If (X, d) is a metric space and $E \subset X$, then **(1 Each)**

i) $\overline{E}$ is closed ;

ii) $E = \overline{E}$ if and only if E is closed;

iii) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

Theorem 24) Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup(E)$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed. **(2 M)**

Theorem 25) If (X, d) is a metric space and $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some subset G of X. **(2 M)**

Theorem 26) If (X, d) is a metric space and $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y. **(2 M)**

Theorem 27) Compact subset of metric spaces are closed. **(4 M)**

Theorem 28) Closed subsets of compact sets are compact. **(4 M)**

Theorem 29) If F is a closed and K is compact , then $F \cap K$ is compact. **(2 M)**

Theorem 30) If a set E in $\mathcal{R}^k$ has one of the following three properties, then it has the other two;

i) E is closed and bounded;

ii) e is compact;

iii) Every infinite subset of E has a limit point in E. **(2 Each)**

Theorem 31) (Weierstrass Theorem) Every bounded infinite subset of $\mathcal{R}^k$ has a limit point in $\mathcal{R}^k$. **(4 M)**

Theorem 32) Let $\{p_n\}$ be a sequence in metric space (X, d)

*i)* $\{p_n\}$ converges to $p \in X$ if and only if for every neighborhood of $p$ contains all but finitely many terms in $\{p_n\}$;

*ii)* If $p \in X$, $p' \in X$ and $\{p_n\}$ converges to both $p$ and $p'$ then $p = p'$ ;

*iii)* If $\{p_n\}$ is convergent then sequence $\{p_n\}$ is bounded;

*iv)* If $E \subset X$ and $p$ is limit point of E then there is a sequence $\{p_n\}$ in E such that $\{p_n\}$ converges to $p$. **(2 Each)**

Theorem 33) Suppose $\{s_n\}$ and $\{t_n\}$ are sequences of real numbers and $s_n \to s$ *and* $t_n \to t$, then

i) $s_n + t_n \to s + t$

ii) $s_n t_n \to s t$

iii) $c\, s_n \to cs$ for any $c \in \mathcal{R}$

iv) $c + s_n \to c + s$ *for any* $c \in \mathcal{R}$.

v) $\frac{1}{s_n} \to \frac{1}{s}$ provided $s_n \neq 0$ (n = 1, 2, ..., ) and $s \neq 0$. **(2 each)**

Theorem 34) Every bounded sequence in $R^k$ contains a convergent subsequence. **(2 M)**

Theorem 35) The subsequential limits of a sequence $\{p_n\}$ in a metric space (X, d) form a closed subset of X. **(4 M)**

Theorem 36) If $\overline{E}$ is the closure of a set E in a metric space X, then diam($\overline{E}$) = diam(E). **(4 M)**

Theorem 37) In any metric space (X, d), every convergent sequence is a Cauchy sequence. In $R$, every Cauchy sequence converges. **(5 M)**

Theorem 38) Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded. **(5 M)**

Theorem 39) If $s_n \leq t_n$ for $n \geq N$, where N is fixed positive integer, then $\underline{\lim}\, s_n \leq \underline{\lim}\, t_n$ *and* $\overline{\lim}\, s_n \leq \overline{\lim}\, t_n$. **(4 M)**

Theorem 40) If $\{s_n\}$ and $\{t_n\}$ are two sequences in $R$, then **(4 Each)**

i) $\underline{\lim}(s_n + t_n) \geq \underline{\lim}\, s_n + \underline{\lim}\, t_n$

ii) $\overline{\lim}(s_n + t_n) \leq \overline{\lim}\, s_n + \overline{\lim}\, t_n$

Theorem 41) Prove the following: **(2 M)**

i) If p > 0, then $\lim\limits_{n \to \infty} \frac{1}{n^p} = 0$

ii) If p > 0, then $\lim\limits_{n \to \infty} \sqrt[n]{p} = 1$

iii) $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$

iv) If p > 0 and α is real, then $\lim\limits_{n \to \infty} \frac{n^{\alpha}}{(1+p)^n} = 0$

v) If $|x| < 1$, then $\lim\limits_{n \to \infty} x^n = 0$

Theorem 42) Series $\sum a_n$ converges if and only if for every $\in > 0$ there is an integer N such that $\left|\sum_{k=n}^{m} a_k\right|$ $if\ \ m \geq n \geq N$ **(4 M)**

Theorem 43) If $\sum a_n$ converges, then $\lim_{n \to \infty} a_n = 0$. **(4 M)**

Theorem 44) A series of nonnegative terms converges if and only if its partial sums form a bounded sequence. **(5 M)**

Theorem 45) Let $\sum a_n$ and $\sum b_n$ be series of non-negative real numbers such that $a_n \leq b_n$ for $n \geq N_0$, where $N_0$ is fixed integer, then

i) if $\sum b_n$ converges then $\sum a_n$ converges

ii) if $\sum a_n$ diverges then $\sum b_n$ diverges. **(6 M)**

Theorem 46) If $0 \leq x < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x \geq 1$, the series is diverges. **(4 M)**

Theorem 47) Suppose $a_1 \geq a_2 \geq . . . \geq 0$. Then the series $\sum a_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k\, a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$ converges. **(4 M)**

Theorem 48) $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. **(4 M)**

Theorem 49) Show that the following: **(2 Each)**

i) If $p > 1$, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges; if $p \leq 1$, the series diverges.

ii) $\sum_{n=3}^{\infty} \frac{1}{n\, l\, og\, n\ \log\log n}$ diverges.

iii) $\sum_{n=3}^{\infty} \frac{1}{n \log n (\log\log n)^2}$ converges.

Theorem 50) Show that $\lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n = e$ **(6 M)**

Theorem 51) Sate and prove Root test. **(5 M)**

Theorem 52) Sate and prove Ratio test. **(5 M)**

Theorem 53) Give the power series $\sum a_n z^n$, if $\alpha = \limsup \sqrt[n]{|a_n|}$, $R = \frac{1}{\alpha}$ then $\sum a_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$. **(5 M)**

Theorem 54) Suppose

a) the partial sums sequence $s_n$ associated with series $\sum a_n$ forms a bounded sequence.

b) $b_0 \geq b_1 \geq b_2 \geq \ldots$;

c) $\lim_{n \to \infty} b_n = 0$

Then $\sum a_n b_n$ converges. **(5 M)**

Theorem 55) A function f of a metric space X into a metric space Y is continuous on X if and only if every open set V in Y, $f^{-1}(V)$ is open in X. **(5 M)**

Theorem 56) Sate and prove fundamental theorem of calculus. **(7 M)**

Theorem 57) State and prove Mean Value Theorem. **(7 M)**

Theorem 58) State and prove Cauchy-Schwarts Inequality. **(7 M)**

Theorem 59) If $f$ and $g$ are defined on [a, b] and are differentiable at a point $x \in$ [a, b] then show that $f + g$ is also differentiable. **(6 M)**

Theorem 60) If f and g are defined on [a, b] and are differentiable at a point x $\in$ [a, b] then show that f.g is also differentiable. **(6 M)**

Theorem 61) Let $f$ and $g$ be the real and differentiable in (a, b). Suppose $g'(x) \neq 0$ for all $x \in$ (a, b), where $-\infty \leq a < b \leq \infty$. Suppose $\lim_{x \to a} \frac{f'(x)}{g'(x)} = A$. If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ then prove that $\lim_{x \to a} \frac{f(x)}{g(x)} = A$. **(7 M)**

Theorem 62) Show that continuous function is Riemann integrable. **(4 M)**

Theorem 63) Prove that if $f$ is a continuous real valued function on closed bounded interval [a, b]. If the maximum value for $f$ is attained at c, where c $\in$ (a, b) and if $f'(c)$ exists, then $f'(c) = 0$. **(8 M)**

Theorem 64) If $f$ and $g$ are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$, then show that $(f \cdot g)$ is also differentiable and

$(f \cdot g)'(x) = g(x) f'(x) + f(x) g'(x)$. **(8 M)**

Theorem 65) Sate and prove Taylor's theorem. **(7 M)**

Theorem 66) Show that a monotone function f(x) bounded on [a, b] is Riemann integrable. **(5 M)**

Theorem 67) If $P_1 \subset P_2$ then show that $U(P_1, f) - L(P_1, f) \geq U(P_2, f) - L(P_2, f)$. State the significance of the result. **(5 M)**

Theorem 68) Show that a bounded monotone function f(x) on [a, b] is Riemann integrable. **(5 M)**

Theorem 69) **(6 M)**

Theorem 70) If $p_2$ is a finer partition of $p_1$ then show that $U(p_1,f) - L(p_1,f) \geq U(p_2,f) - L(p_2,f)$. State the significance of the result. **(5 M)**

Theorem 71) State and prove the necessary and sufficient condition when the Riemann integral is $\int_a^b f(x)dx$ exists. **(5 M)**

**Q.3** Define metric space. Check whether the following is metric or not. **(4 Each)**

i) $d(x,y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$

ii) $d(x,y) = |x - 2y|\ \ for\ x, y \in \Re$

iii) $d(x,y) = \dfrac{|x-y|}{1+|x-y|}$

iv) $d(x,y) = \max\{|x|,|y|\}\quad \forall x, y \in$ □

v) $d(x,y) = |x^2 - y^2|,\quad \forall x, y \in$ □

vi) $d(x,y) = |x - y|\quad \forall x, y \in$ □

vii) $d(x,y) = \sqrt{|x-y|}\quad for\ x, y \in R$

viii) $d(x,y) = (x-y)^2\quad for\ x, y \in R$

ix) In $R^n$, with $\underline{x} = (x_1, x_2, \ldots, x_n)$ and $\underline{y} = (y_1, y_2, \ldots, y_n)$, define,
$d(\underline{x}, \underline{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$.

**Q. 4.** Suppose that $d_1(x, y)$ and $d_2(x, y)$ are two metrics on a non-empty set $S$. Further $0 < \lambda < 1$ is a real number. Examine whether the following functions are metrics.

**(3 Each)**

i. $d(x, y) = d_1(x, y) + d_2(x, y)$

ii. $d(x,y) = d_1(x,y) * d_2(x,y)$

iii. $d(x, y) = \max\{d_1(x, y), d_2(x, y)\}$

iv. $d(x, y) = \min\{d_1(x, y), d_2(x, y)\}$

v. $d(x, y) = \lambda\, d_1(x, y) + (1 - \lambda)\, d_2(x, y)$

vi. $d(x, y) = \min\{d_1(x, y), 1\}$

**Q. 5.** Obtain the radius of convergence for the following series. **(3 each)**

i) $\sum \dfrac{n^n}{n!} z^n$

ii) $\sum \dfrac{n^3}{3^n} z^{2n}$

iii) $\sum \dfrac{2^n}{n!} z^n$

iv) $\sum \frac{n^3}{3^n} z^n$  v) $\sum \frac{2^n x^n}{n^2}$  vi) $\sum \frac{(-1)^n n^2 x^n}{5^n \sqrt{n^5}}$

**Q.6** Suppose $\{a_n\}$ and $\{b_n\}$ are sequence of real numbers and $a_n \to a$ and $b_n \to b$ then show that $xa_n + yb_n \to xa + yb$ where x and y are any real constants. **(6)**

**Q. 7** Discus the convergence of following series: **(3 each)**

i) $\sum \frac{1}{n\sqrt{n}}$

ii) $\sum \frac{n+1}{3(n+3)}$

iii) $\sum \frac{(-1)^n}{n^2}$

iv) $\sum \frac{1}{n^2+1}$

v) $\sum \cos \frac{\pi}{n}$

vi) $\sum \frac{1+n}{2+3n^2}$

**Q. 8** State and prove Cauchy criterion for convergence of series. **(8)**

**Q. 9** Show that the following sets are uncountable sets: **(4 each)**

i) Set of all sequences whose elements are the digits zero and one.

ii) Set of real numbers in [0, 1]

**Q. 10** Define Nested Chain. State and prove Nested Chain theorem. **(8)**

**Q. 11** Check whether following is field or not **(4 each)**

i) $(\mathbb{Q} ; + ; .)$

ii) F = {-1, 0, 1}

iii) $(\mathbb{R}; +; .)$

**Q. 12** Check whether the following sets are countable set or uncountable set: **(2 each)**

i) Set of integers

ii) Set of rational numbers

iii) Set of real numbers in [0, 1]

iv) Set A = $\left\{0, \pm \frac{1}{n}, n = 1, 2, 3, \ldots\right\}$

v) Set of all non-positive integers

vi) Set of all positive multiples of five

vii) Set of all prime numbers

**Q. 13** For any two real sequence $\{a_n\}$ and $\{b_n\}$, then prove that **(4)**

i) $\lim_{n\to\infty} \inf a_n + \lim_{n\to\infty} \inf b_n \leq \lim_{n\to\infty} \inf (a_n + b_n)$

ii) $\lim_{n\to\infty} \sup (a_n + b_n) \leq \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \sup b_n$

iii) $\lim \sup(-a_n) = -\lim \inf a_n$

iv) $\lim \inf(-a_n) = -\lim \sup a_n$

**Q. 14** If $\sum a_n$ is convergent then show that $a_n \to 0$ as $n \to \infty$. What does about converse of this result? Justify your answer. **(6)**

**Q. 15** Let $A_1, A_2, \ldots$ be subsets of a metric space. And $B_n = \bigcup_{i=1}^{n} A_i$, Prove that $\overline{B}_n = \bigcup_{i=1}^{n} \overline{A}_i$ **(7)**

**Q. 16** If $\sum a_n$ is converges, $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges. **(7)**

**Q. 17** If $r$ is rational ($r \neq 0$) and $x$ is irrational, then prove that $r + x$ and $rx$ are irrational. Hence prove that $\sqrt{12}$ is irrational number. **(6)**

**Q. 18** Prove or disprove the following sentence: **(2 each)**

i) Countable intersection of open sets is open.

ii) Set of Integers I = {0, ± 1, ± 2, ± 3, . . . } is countable set.

iii) If series $\sum a_n$ is convergent then series $\sum |a_n|$ is also convergent.

iv) Every bounded sequence is convergent sequence.

v) Set of rational numbers in [0, 1] is countable.

vi) Countable Union of Closed set is Closed.

vii) Every convergent sequence is bounded sequence.

viii) If series $\sum a_n$ is convergent then series $\sum |a_n|$ is also convergent.

ix) Every neighborhood is an open set.

x) Set of irrational numbers is uncountable.

**Q.19** Solve the following examples: **(2 each)**

i) Find limit superior and limit inferior of $t_n = \sin\left[\pi\left(n - \frac{1}{2}\right)\right]$

ii) Find *lub* and *glb* of the set $\left\{1 + \frac{1}{n} \mid n \geq 1\right\}$

iii) Find limit inferior and limit superior of the sequence $\{a_n\} = \left\{\frac{n+1}{n}\right\}$.

iv) Discuss convergence of the series $\sum \frac{1+n}{2+3n^2}$.

v) If F = {-1, 0, 1}; then show that (F, +, ·) is a field.

vi) If $x = (x_1, x_2), y = (y_1, y_2)$ are any two points in $R^2$ and define $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ then show that d is metric in $R^2$.

vii) Find limit superior and limit inferior of $t_n = (-1)^n \left(1 + \frac{1}{n}\right)$

viii) Find *lub* and *glb* of the set $\{x \in R \mid x^2 - 5x + 6 < 0\}$

ix) Discuss convergence of the series $\sum \frac{1}{2+3n^2}$.

x) If $I$ is set of integers, then check whether ($I$, +, ·) is a field.

xi) Show that d is metric, if $x=(x_1,x_2), y=(y_1,y_2)$ are any two points in $R^2$, define $d(x,y)=\max\{|x_1-y_1|,|x_2-y_2|\}$

xii) Find least upper bound and greatest lower bound of the set {1, ½, 1/3, ¼, …}.

xiii) Discus convergence of the sequence $\left\{\frac{(-1)^n}{n}\right\}$

**Q.20.** Give an example of a set satisfying the condition in each of the following (Specify metric space in each case): **(2 each)**

i) Set is neither closed nor open set.

ii) Bounded set which is closed set as well as open set.

iii) Unbounded set which is closed set as well as open set

iv) Bounded set which is closed but not open set

v) Unbounded set which is Closed set but not Open set

**Q. 21.** Give an example of each of the following. Clearly state the metric space in each case. **(2 each)**

i) A set with no interior point.

ii) A set with exactly one interior point.

iii) A set with $n$ interior points, for $n > 1$.

iv) A set with countably many interior points.

v) A set with uncountably many interior points.

vi) A set with no limit point

vii) A set with exactly one limit point

viii) A set with $n$ limit points, for $n > 1$

ix) A set with countably many limit points

x) A set with uncountably many limit points

**Q. 22.** Examine whether the following sets are open/closed. **(2 each)**

i) $E_1 = \{y \mid d(x,y) < \varepsilon\}$

ii) $E_2 = \{y \mid d(x,y) \le \varepsilon\}$

iii) $E_3 = \{y \mid d(x,y) > \varepsilon\}$

iv) $E_4 = \{y \mid d(x,y) \ge \varepsilon\}$

v) $E_5 = \{y \mid d(x,y) = \varepsilon\}$

**Q. 23.** Give an example of each of the following. **(2 each)**

i) A relation/mapping which is not a function

ii) A function which is neither one-one nor onto

iii) A function which is one-one but not onto

iv) A function which is onto but not one-one

v) A function which is both one-one as well as onto

**Q. 24.** Give an example of a set, with $l$ = inf A and $u$ =sup A satisfying the following: **(2 each)**

i) $l \in$ A but $u \notin$A

ii) $l \notin$ A and $u \notin$A

iii) $l \notin$ A but $u \in$A

iv) $l \in$ A and $u \in$A

v) $l = u$

**Q. 25.** Let $A$ be a nonempty set of real numbers which is bounded below. Let -$A$ be the set of all numbers of the form -$x$, where $x \in A$, then prove that $inf\, A$ = - $sup$(-$A$). **(2)**

**Q. 26.** If $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$, for n = 1, 2, 3, …. Prove that $2 < \lim_{n\to\infty} s_n < 3$. **(5 M)**

**Q. 27.** If $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ for n ≥ 1. Prove that $\{s_n\}$ is convergent sequence. **(5 M)**

**Q. 28.** Suppose P and Q are the partitions of (0, 1) where P = {0, 0.25, 0.50, 0.75, 1} and Q = { 0, 0.33, 0.67, 1}.

i) Find || P|| and ||Q||

ii) State with justification true or false: P is finer partition of Q. **(2+2)**

**Q. 29.** Evaluate the limits of Upper Riemann Sum and Lower Riemann Sum as the norm partition tends to zero. Hence find $\int_0^b x^2 dx$ . **(6 M)**

**Q. 30.** Evaluate $\int_0^b x\, d[x]$, taking limit of Riemann –Stieltejes sum as the norm of partition tends to zero. **(6 M)**

**Q. 31.**