Anekant Education Society's

TuljaramChaturchand College, Baramati

Department of Mathematics

Class: M.Sc -I

Question Bank of Group Theory [MAT4103]

1. Define Group. Give one example of a group.
2. Prove the uniqueness of identity element. Also shown that every element in a Group has a unique inverse.
3. Find the inverse of element $\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ in GL(2,Z_{11}).
4. For any element *a* and *b* of a group *G*, show that $(a^{-1}ba)^{-n}=a^{-1}b^{n}a$, for any integer *n.*
5. Prove that group *G* is abelian if and only if $(ab)^{-1}=a^{-1}b^{-1}$ for all *a, b* in *G.*
6. Prove that if $(ab)^2=a^2b^2$ in a group, then *ab=ba.*
7. Let *G*be a finite group and *H* be a subset of *G.* Show that *H* is a subgroup of *G* if *H* is closed under the operations on *G.*
8. Let *G* be a group and let *a* be any element of *G*, then show that $<a>$ is a subgroup of *G.*
9. Define centre of a Group.
10. Show that centre of a Group *G* is a subgroup of *G.*
11. Define centralizer of an element in a group.
12. Prove that in any group, an element and its inverse have the same order.
13. How many subgroups of order 4 does D4 have?
14. In the group Z12, find |a|, |b|, and |a + b| for each case.
    a. a = 6, b = 2
    b. a = 3, b = 8
    c. a = 5, b = 4
15. Show that if a is an element of a group G, then $|a| \neq |G|$.
16. Show that U(14) = $<3> = <5>$.
17. Let G be a group, and let *a* be an element of *G*. Prove that C(a) = C(a^{-1}).
18. Suppose a group contains elements a and b such that |a| = 4, |b| = 2, and $a^3b = ba$. Find |ab|.
19. Let a be a group element of order n, and suppose that d is a positive divisor of n. Prove that $|a^d| = n/d$.
20. Consider the elements $A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$and $B=\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$from SL(2, R). Find |A|, |B|, and |AB|.
21. Prove that a group of even order must have an element of order 2.
22. Prove that, every permutation of a finite set can be written as a product of disjoint cycles.
23. Define the terms: a) Symmetric group b) Alternating Group.
24. Prove that , two disjoint cycles commute in permutation group.
25. Prove that, the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.
26. Prove that every permutation in S_n, n>1, is a product of 2-cycles.
27. Prove that the set of even permutation in S_n forms a subgroup of S_n.
28. Prove that , for n>1 A_n has order n!/2.
    Determine whether the following permutations are even or odd.
    a)(1 3 5) b)(1 2)(1 3 4)(1 5 2)
29. Prove that A_n is a subgroup of S_n.
30. Show that A_8 contains an element of order 15.
31. Let $\beta \in S_7$ and suppose β^4=(2 1 4 3 5 6 7). Find β.
32. In S_3, find elements α and β so that $|\alpha|=2$, $|\beta|=2$ and $|\alpha\beta|=3$.
33. Let G be the set of all permutations of the positive integers. Let H be the subset of elements of G that can be expressed as a product of a finite number of cycles. Prove that H is a subgroup of G.

34. Show that if H is a subgroup of S_n, then either every member of H is an even permutation or exactly half of them are even.
35. Find order of the permutations. a)$\begin{bmatrix} 1 & 23 & 45 & 6 \\ 2 & 15 & 46 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 23 & 45 & 67 \\ 7 & 61 & 23 & 45 \end{bmatrix}$
36. Prove that S_n is non-Abelian for all $n \geq 3$.
37. Let α and β belongs to S_n. Prove that $\beta\alpha\beta^{-1}$ and α are both even or odd.
38. Prove that, the order of an element of a direct product of a finite number of finite groups is a least common multiple of the orders of the components of the elements.
39. Let G and H be finite cyclic groups. Then prove that $G \oplus H$ is cyclic if and only if |G| and |H| are relatively prime.
40. Determine all elements of order 5 in $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$.
41. Determine the number of cyclic subgroup of order 10 in $\mathbb{Z}_{100} \oplus \mathbb{Z}_{25}$.
42. Prove that $G_1 \oplus G_2$ is isomorphic to $G_2 \oplus G_1$.
43. Is $\mathbb{Z} \oplus \mathbb{Z}$ a cyclic group? Justify.
44. Define the term : a)Normal Subgroup. b) Factor Group.
45. Let G be a group and let Z(G) be the cente of G. If G/Z(G) is cyclic then prove that G is Abelian.
46. For any group G, Prove that G/Z(G) is isomorphic to Inn(G).
47. Let G be a finite Abelian group and let p be a prime that divides the order of G. Then prove that G has an element of order p.
48. Prove that A_n is a normal subgroup of S_n.
49. Prove that every factor group of cyclic group is cyclic.Is converse true? Justify.
50. Prove that SL(2, $\mathbb{R}$) is a normal subgroup of GL(n,$\mathbb{R}$)
51. Find order of 5+<6> in the factor group $\mathbb{Z}_{18}$/<6>.
52. If N and M are normal subgroups of G, prove that NM is also a normal subgroup of G.
53. Determine all Automorphism of D_n.
54. Show that intersection of two normal subgroups is again a normal subgroup.
55. Let ϕ be a homomorphism from a group G to a group $\bar{G}$ and let H be a subgroup of G. Then prove that a) ϕ(H) is a subgroup of $\bar{G}$
    b) If H is abelian then ϕ(H) is abelian.
56. State and prove all three Sylow's theorems.
57. Determine all groups of order 99 .
58. Find all abelian groups of order 360 upto isomorphism.
59. Let G be a finite Abelian group of order $p^n m$ where p is a prime that does not divide m. Then prove that $G=H \times K$, where $H=\{x \in G / x^{p^m=e}\}$ and $K=\{x \in G / x^m=e\}$. Moreover $|H|=p^n$.
60. Let G be an abelian group of prime order and let a be an element of maximal order in G. then prove that G can be written as <a>×K.
61. Let G={1,7,17,23,49,55,65,71} under multiplication modulo 96.
62. Express G as an external and internal direct product of cyclic group.
63. Define the terms : Cyclic Group
64. .Let G be a group and let a be an element of order n in G. If $a^k=e$ then prove that n divides k.
65. Let G=<a> be a cyclic group of order n. Then , prove that G =<a^k> if and only if gcd(k,n)=1.
66. State and prove Fundamental Theorem of Cyclic Groups..
67. Find all generators of a) $\mathbb{Z}_8$ b) $\mathbb{Z}_{20}$.
68. Let G be a group and let a∈ G. Prove that <a^{-1}>=<a>.
69. Give an example of a noncyclic group, all of whose proper subgroups are cyclic.
70. If |a|=n. Show that <a^k>=<$a^{\gcd(n,k)}$> and that $|a^k|$= n/gcd(n,k).
71. 9 Suppose a and b belong to a group, ahasodd , and $aba^{-1}=b^{-1}$. Show that $b^2=e$.
72. Determine the subgroup lattice diagram for the following groups.
    a)$\mathbb{Z}_8$ b)$\mathbb{Z}_{20}$ c) U(12)
73. List all elements of $\mathbb{Z}_{40}$ that have order 10.
74. Find Automorphisms of Z, where Z is set of integers.

75. Show that U(8) is not isomorphic to U(12).
76. State and prove Cayley's theorem.
77. Is (Q,+) is isomorphic to $(Q^*,\cdot)$,where Q is set of rational numbers and Q^* is set of all nonzero rational numbers.
78. Suppose ∅ is an isomorphism from a group G onto a group $\bar{G}$.Then prove that , ∅carries identity of G to the identity of $\bar{G}$.
79. Suppose ∅ is an isomorphism from a group G onto a group $\bar{G}$.Then prove that , G is abelian if and only if $\bar{G}$is abelian.
80. Suppose ∅ is an isomorphism from a group G onto a group $\bar{G}$.Then prove that , G is cyclic if and only if $\bar{G}$ is cyclic. .
81. Define Automorphism and inner automorphism of group.
82. Prove that Q under addition is not isomorphic to R^+ under multiplication.
83. Determine innerautomorphisms of D_4.
84. For every positive integer n, Aut(Z_n) is isomorphic to U(n).
85. Find two groups G and H such that G$\not\approx H$ but Aut(G) ≈ Aut(H).
86. State and prove Lagranges theorem.
87. Is converse of lagranges theorem true?
88. Prove that a group of prime order is cyclic.
89. Define stabilizer of a point.
90. State and prove orbit – stabilizer theorem.
91. Find all left cosets of {1,11} in U(30).
92. Suppose H and K are subgroups of a group G.If$|H| = 12\ \ and\ |K| = 35$ then find $|H \cap K|$.
93. Let $|a| = 30$.Hoe many left cosets of $\langle a^4\rangle$ in $\langle a\rangle$ are there? List them.
94. Let G={(1),(132)(465)(78),(132)(465),(123)(456),(123)(456)(78),(78)},then find orbit of a point 4 ,also find stabilizer of point 7.
95. Let G be a group of order 60.What are the possible orders for the subgroups of G.
96. Let ∅ be a group homomorphism from G to $\bar{G}$.Then prove that Ker∅ is a normal subgroup of G.
97. Suppose ∅ is an homomorphism from a group G to a group $\bar{G}$.Then prove that , ∅ carries identity of G to the identity of $\bar{G}$.
98. Suppose ∅ is an homomorphism from a group G to a group $\bar{G}$.Then prove that ,for g is an element of G with $|g| = n\ , then\ |\emptyset(g)|$ divides n.
99. State and prove first isomorphism theorem.
100. How many homomorphisms are there Z_{20} onto $Z_{8.}$
101. Determine all homomorphisms from Z_n to itself.
102. Find homomorphism ∅ from U(30) to U(30) with kernel {1,11} and $\emptyset(7) = 7$.
103. Find homomorphism ∅ from U(40) to U(40) with kernel {1,9,17,33} and $\emptyset(11) = 1$