Anekant Education Society's

TULJARAM CHATURCHAND COLLEGE, BARAMATI

Department of Mathematics

F.Y.B.Sc. Question Bank

Paper–I: *Algebra*

Paper Code-MAT1101

Answer in One Sentence (or in 2 – 3 lines) (2 marks questions)

1. State the principle of mathematical induction.
2. Define Cartesian product of sets.
3. True or false: If $|A| = 5$, then $|P(A)| = 25$.
4. Define range of a function.
5. Define a one-one function.
6. Let $f: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ define by $f(x) = x^2$. Does f injective?
7. Define onto function.
8. Let $f: \boldsymbol{R} \to \boldsymbol{R}$ define by $f(x) = x^2$. Does f surjective?
9. Let $f: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ define by $f(x) = x^2$. Does f bijective?
10. Let $f: \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = x^2$. Find $f^{-1}([4, 16])$.
11. Define composition of functions.
12. Define equivalence relation on a set.
13. Define partition of a set.
14. True or false: If $|A| = 3$ and $|B| = 5$, then total number of relations from A to B is 15.
15. Define greatest common divisor of integers.
16. Show that there are no two integers a, b such that $(a, b) = 3$ and $a + b = 100$.
17. If $b = aq + r$, $0 \leq r < |a|$, then prove that $(a, b) = (a, r)$.
18. True or false: A number is divisible by 4 if and only if last digit of the number is divisible by 4.
19. Find all primes which divide $50!$.
20. If p is prime and $a^2 \equiv b^2 (mod\ p)$, then show that either $p|(a + b)$ or $p|(a - b)$.
21. In $\mathbb{Z}_{12}$, calculate *i*) $(\overline{2} \cdot \overline{9} + \overline{1})^{-1}$, *ii*) $-\overline{5} \cdot (\overline{4} + \overline{5})$.
22. Show that $z = 1 + i$ satisfies the equation $z^2 - 2z + 2 = 0$.
23. Show that $Re(iz) = -Im(z)$.
24. Show that $Im(iz) = Re(z)$.
25. Compute the modulus and principal argument of $\frac{1}{1+i}$.
26. Express $-1 + i$ in polar form.
27. State De Moivre's theorem.
28. Define row echelon form.
29. Define rank of a matrix.
30. Write augmented matrix for the following system of linear equations
    $x + 2y - 3z + 4w = 2$, $2x + 5y - 2z + w = 1$, $5x + 12y - 7z + 6w = 7$.

31. Find the rank of $\begin{bmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 1 & 0 & 1 \end{bmatrix}$.

32. Define eigenvalues and eigenvectors of a matrix.

33. Find characteristics polynomial of $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

34. Find eigenvalues of $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

35. Find eigenvalues of $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

Short Answer Questions (4 marks questions)

1. Prove that the sum of first n positive integers is $\frac{n(n+1)}{2}$.

2. Prove that $3|(n^3 + 2n)$, whenever n is a positive integer.

3. Show that $n^2 < 2^n$ for all positive integers $n \geq 5$.

4. Prove that $n! > 2^n$ for every integer $n \geq 4$.

5. Let A and B be sets. Show that $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

6. Let A and B be sets. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

7. Let A, B and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$.

8. Suppose $B \subseteq C$. Prove that for any set A,

   i) $A \cup B \subseteq A \cup C$.
   ii) $A \cap B \subseteq A \cap C$.

9. Let f be a function from A to B. Let S and T be subsets of B. Show that

$$f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T).$$

10. Let f be a function from A to B. Let S and T be subsets of B. Show that

$$f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T).$$

11. If $f: \boldsymbol{R} \to \boldsymbol{R}$ is defined by $f(x) = 2x + 3$, then show that $f$is bijection and hence find f^{-1}.

12. Let $f: [0, 1] \to [a, b]$, $(a < b)$ defined as $f(x) = bx + a(1 - x)$. Show that f is a bijection.

13. Let $A = \{a, b, c\}$ and $B = \{1, 2\}$. List all possible functions from A to B.

14. Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be defined as $f(x) = x^3 + 2$, and $g: \boldsymbol{R} \to \boldsymbol{R}$ be defined as $g(x) = \sqrt[3]{x}$. Find $g \circ f$ and $f \circ g$. Are they same?

15. Is the following relation R on a set $A = \{0, 1, 2, 3\}$ equivalence relation?

    $R = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3)\}$.

16. Let R be the relation on $\mathbb{Z} \times \mathbb{Z}$ given by $(x, y)R(s, t)$ if and only if $xt = ys$. Prove that R is an equivalence relation.

17. For positive integer $n > 1$, define a relation $\sim$ on $\mathbb{Z}$ by $x \sim y$ if and only if $n|(x - y)$. Show that $\sim$ is an equivalence relation on $\mathbb{Z}$.

18. Prove that equivalence classes are either identical or disjoint.

19. If X be a nonempty set, then prove that the equivalence classes of an equivalence relation on X induces a partition of X.

20. If $a|b$ and $a|c$ then prove that $a|(bx + cy)$ for any integers x and y.

21. Find all integers n such that $n^2 + 1$ is divisible by $n + 1$.

22. If a, b are integers, not both zero, then prove that there are integers x_0 and y_0 such that

$$(a, b) = ax_0 + by_0.$$

23. If $(a, b) = d$, then prove that $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

24. Let $a > b$ and $(a, b) = 1$. Show that $(a + b, a - b)$ is either 1 or 2.

25. Prove that if p is a prime and a, b are integers such that $p|ab$, then either $p|a$ or $p|b$.

26. Show that there are infinitely many primes of the form $4n + 3$.

27. Show that there are infinitely many primes of the form $6n - 1$.

28. Show that $\sqrt{2}$ is not rational number.

29. Show that $\sqrt{3}$ is not rational number.

30. If $a \equiv b(mod\ n)$ and $c \equiv d(mod\ n)$, then prove that

    i) $(a + c) \equiv (b + d)(mod\ n)$.

    ii) $ac \equiv bd(mod\ n)$.

31. If $ax \equiv bx(mod\ n)$ and $(x, n) = 1$, then prove that $a \equiv b(mod\ n)$.

32. Prove that there are precisely n distinct residue classes modulo n.

33. Prepare addition and multiplication table for $\mathbb{Z}_6$.

34. Prepare addition and multiplication table for $\mathbb{Z}_5$.

35. State and prove Fermat's theorem.

36. Show that 41 divides $2^{20} - 1$.

37. Find the remainder of 7^{486} when divided by 13.

38. Find the remainder of $7^{361} + 7^{362}$ when divided by 11.

39. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers, then prove that

i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$,

ii) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.

40. Find $Re\ z$ and $Im\ z$, where $z = \left(\frac{2+i}{3-2i}\right)^2$.

41. Find two complex numbers whose sum is 4 and whose product is 8.

42. Let z_1 and z_2 are complex numbers and $z_2 \neq 0$, then prove that

i) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

43. For $z_1, z_2 \in \mathbb{C}$, prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.

44. For $z_1, z_2 \in \mathbb{C}$, prove that $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$.

45. Compute the modulus and principal argument of

i) $i^7 + i^{10}$

ii) $\frac{(1+i)^3}{(1-i)^2}$.

46. Show that the identity $|z - 1| = |z + 1|$ represents imaginary axis.

47. If $|z_1| = |z_2| = |z_3| = 5$ and $z_1 + z_2 + z_2 = 0$, then prove that $\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0$.

48. State and prove De Moivre's theorem.

49. Find fifth root of -1.

50. Prove that $(-1 + i)^7 = -8(1 + i)$.

51. Solve the equation $x^7 - x^4 - x^3 + 1 = 0$.

52. Evaluate $(1 + i)^{\frac{1}{3}}$.

53. Solve the equation $x^6 + i = 0$.

54. Solve the following system by Gauss elimination method

$$x + y + 2z = 9,\ 2x + 4y - 3z = 1,\ 3x + 6y - 5z = 0.$$

55. Solve the following system by Gauss elimination method

$$x + y + 2z = 8,\ -x - 2y + 3z = 1,\ 3x - 7y + 4z = 10.$$

56. Solve the following system by Gauss elimination method

$$2x + y + 5z + w = 5,\ x + y - 3z - 4w = -1,\ 3x + 6y - 2z + w = 8,$$

$$2x + 2y + 2z - 3w = 2.$$

57. Solve the following system by Gauss Jordan method

$$2x + z = 4,\ x - 2y + 2z = 7,\ 3x + 2y = 1.$$

58. Solve the following system by Gauss Jordan method

$$2x - y - 3z = 0,\ -x + 2y - 3z = 0,\ x + y + 4z = 0.$$

59. Solve the following system by Gauss Jordan method

$$2x + y + 5z + w = 5,\ x + y - 3z - 4w = -1,\ 3x + 6y - 2z + w = 8,$$

$$2x + 2y + 2z - 3w = 2.$$

60. Show that the following system is inconsistent

$$x + y + z = 3,\ 2x - y + 3z = 2,\ 3x - 2y + z = 4,\ 4x + y + 5z = 2.$$

61. Show that the following system have infinitely many solutions

$$2x + z = 4,\ x - 2y + 2z = 7,\ 3x + 2y = 1.$$

62. Find the value of λ and μ so that the system given below admits

a) Unique solution b) No solution c) Infinitely many solutions

$$x + y + z = 6,\ x + 2y + 3z = 10,\ x + 2y + \lambda z = \mu.$$

63. Find the value of λ and μ so that the system given below admits

b) Unique solution b) No solution c) Infinitely many solutions

$$2x + 3y + 5z = 9,\ 7x + 3y - 2z = 8,\ 2x + 3y + \lambda z = \mu.$$

64. Find eigenvalues and eigenvectors of $\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$.

65. Find eigenvalues and eigenvectors of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$.

66. Find eigenvalues and eigenvectors of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

67. Find eigenvalues and eigenvectors of $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$.

68. Verify Cayley Hamilton theorem and use it to find A^{-1}, if it exists.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}.$$

69. Verify Cayley Hamilton theorem and use it to find A^{-1}, if it exists.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$

70. Verify Cayley Hamilton theorem and use it to find A^{-1}, if it exists.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

Long Answer Questions (8 marks questions)

1. Find g.c.d. of 1769 and 2378, and express it in the form
$$(1769, 2378) = 1769x_0 + 2378y_0 \text{ for some } x_0, y_0 \in \mathbb{Z}.$$
2. Find g.c.d. of 3997 and 2947, and express it in the form
$$(3997, 2947) = 3997x_0 + 2947y_0 \text{ for some } x_0, y_0 \in \mathbb{Z}.$$
3. Find g.c.d. of 7234 and 3476, and express it in the form
$$(7234, 3476) = 7234x_0 + 3476y_0 \text{ for some } x_0, y_0 \in \mathbb{Z}.$$
4. Find g.c.d. of 5291 and 4514, and express it in the form
$$(5291, 4514) = 5291x_0 + 4514y_0 \text{ for some } x_0, y_0 \in \mathbb{Z}.$$
5. Show that 4999 and 1109 are relatively prime. Also find x_0 and y_0 such that
$$1 = 4999x_0 + 1109y_0.$$