Anekant Eduction Society's

Tuljaram Chaturchand College,Baramati

Department of Mathematics
Question Bank
Class:F.Y.B.Sc.
Title of Paper:Calculus-II
Subject Code:MAT1202

Short Answer Questions

1. Define Continuous Function.
2. Let $f(x) = x^2 sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$. Prove that f is continuous at 0.
3. If f is continuous at x_0 and g is continuous at $f(x_0)$, then prove that the composite function $g \circ f$ is continuous at x_0.
4. Prove that $|x|$ is continuous function $\mathbb{R}$.
5. Prove that $\sqrt{x}$ is continuous on the domain $[0, \infty)$.
6. Prove that if $m \in \mathbb{N}$, then the function$f(x) = x^m$ is continuous on $\mathbb{R}$.
7. Prove that every polynomial function $p(x) = a_0+a_1x+a_2x^2+....+a_nx^n$ is continuous on $\mathbb{R}$.
8. Define the term Rational Function.
9. Prove that every rational function is continuous.
10. Define the term signum function.
11. Let $f(x) = \sqrt{4-x}$ for $x \leq 4$ and $g(x) = x^2$ for all $x \in \mathbb{R}$.
    (a) Give the domains of $f + g, fg, f \circ g$ and $g \circ f$.
    (b) Are the functions $f \circ g$ and $g \circ f$?
12. Let $f(x) = 4$ for $x \geq 0$, $f(x) = 0$ for $x < 0$, and $g(x) = x^2$ for all x.
    (a) Determine the following functions : $f + g, fg, f \circ g, g \circ f$.Also specify their domains.
    (b) Which of the functions $f, g, f + g, fg, f \circ g, g \circ f$ is continuous?
13. Let $f(x) = 1$ for rational numbers x and $f(x) = 0$ for irrational numbers. Show that f is discontinuous at every x in $\mathbb{R}$.
14. Let $f(x) = x$ for rational numbers x and $f(x) = 0$ for irrational numbers. Show that h is continuous at $x = 0$ and at no other point.
15. Define the term Bounded function.
16. Prove $x = cosx$ for some x in $(0, \frac{\pi}{2})$.
17. Prove $xe^x = 2$ for some x in $(0, 1)$.

18. Show that if $y > 0$ and $m \in \mathbb{N}$, then y has a positive mth root.

19. Define the term Strictly increasing function and Strictly decreasing function.

20. Let g be a strictly increasing function on an interval J such that $g(J)$ is an interval I.Then prove that g is continuous on J.

21. Let f be a one-to-one continuous function on an interval I. then prove that f is either strictly increasing or strictly decreasing.

22. Let f be a continuous real valued function on a closed interval $[a, b]$.Show that if $-f$ assumes its maximum at $x_0 \in [a, b]$, then f assumes its minimum at x_0.

23. Use calculus to find the maximum and minimum of $f(x) = x^3 - 6x^2 + 9x + 1$ on $[0, 5)$.

24. Show that $f(x) = \frac{1}{x^2}$ is uniformely continuous on the set of the form $[a, \infty)$ where $a > 0$.

25. Show that $f(x) = \frac{1}{x^2}$ is not uniformely continuous on $(0, \infty)$ and $(0, 1)$.

26. Define the term Limit of a function

27. Find $\lim_{x \to 2} \frac{x^2-4}{x-2}$.

28. Find $\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1}$.

29. Let f_1 an f_2 be functions for which the limits $L_1 = \lim_{x \to a} f_1(x)$ and $L_2 = \lim_{x \to a} f_2(x)$ exist and finite then prove that,
    (a) $\lim_{x \to a}(f_1 + f_2)(x) = L_1 + L_2$.
    (b) $\lim_{x \to a}(f_1 f_2)(x) = L_1 L_2$.
    (c) $\lim_{x \to a}(\frac{f_1}{f_2})(x) = \frac{L_1}{L_2}$ provided $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in S$.

30. If limit of a function exist then prove that it must be unique.

31. Define right hand limit and left hand limit of a function.

32. Find $\lim_{x \to a} \frac{x^3-a^3}{x-a}$.

33. Find $\lim_{x \to 0} \frac{x}{|x|}$ if exist.

34. Define the term Differentiability of a function.

35. Find derivative of following functions at given points.
    (a) $g(x) = x^2$ for $x = 2$.
    (b) $h(x) = \sqrt{x}$ for $x = 1$.
    (c) $f(x) = x^3$ at $x = 2$.
    (d) $f(x) = x^2 cosx$ at $x = 0$.
    (e) $r(x) = \frac{3x+4}{2x-1}$ at $x = 1$.

36. For each of the following functions defined on $\mathbb{R}$, give the set of points at which it is not differentiable.
    (a) $e^{|x|}$.

(b) $|x| + |x-1|$.
(c) $|x^2 - 1|$.
(d) $|sinx|$.
(e) $sin|x|$.

37. Show that $x < tanx$ for all $x \in (0, \frac{\pi}{2})$

38. Show that $\frac{x}{sinx}$ is strictly increasing function on $(0, \frac{\pi}{2})$.

39. Show that $sinx \leq x$ for all $x \geq 0$.

40. Show that $ex \leq e^x$ for all $x \in \mathbb{R}$.

41. Let f be defined on $\mathbb{R}$, and suppose $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

42. Prove that $|cosx - cosy| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

43. Check the Mean Value Theorem for the following functions.
    (a) x^2 on $[-1, 2]$
    (b) $sinx$ on $[0, \pi]$.
    (c) $|x|$ on $[-1, 2]$.
    (d) $\frac{1}{x}$ on $[-1, 1]$.

44. Find $\lim_{x\to 0} \frac{sinx}{x}$.

45. Find $\lim_{x\to 0} \frac{cosx-1}{x^2}$.

46. Find $\lim_{x\to\infty} \frac{x^2}{e^{3x}}$.

47. Find $\lim_{x\to\infty} x^{1/x}$.

48. Find $\lim_{x\to 0^+} x^x$.

49. Find the following limits if exist.
    (a) Find $\lim_{x\to 0} \frac{x^3}{sinx-x}$.
    (b) Find $\lim_{x\to 0} \frac{e^{2x}-cosx}{x}$.
    (c) Find $\lim_{x\to\infty} \frac{x^3}{e^{2x}}$.
    (d) Find $\lim_{x\to\infty} \frac{x-sinx}{x}$.

50. Define Taylor series for a function f about c.

Long Answer Questions:

1. Let $f(x) = 2x^2 + 1$ for $x \in \mathbb{R}$. Prove that f is continuous on $\mathbb{R}$ by
    (a) Using the definition.
    (b) Using the $\epsilon - \delta$ definition.

2. Let f is a real valued function with dom$(f) \subseteq \mathbb{R}$. If f is continuous at x_0 in dom(f), then prove that $|f|$ and kf, $k \in \mathbb{R}$ are continuous at x_0.

3. Let f and g be real valued functions that are continuous at x_0 in $\mathbb{R}$. Then prove that, $f + g$ is continuous at x_0.

4. Let f and g be real valued functions that are continuous at x_0 in $\mathbb{R}$. Then prove that, fg is continuous at x_0.

5. Let f and g be real valued functions that are continuous at x_0 in $\mathbb{R}$. Then prove that, f/g is continuous at x_0 if $g(x_0) \neq 0$.

6. Let f and g is continuous at x_0 in $\mathbb{R}$. Prove that $\max(f, g)$ is continuous at x_0.

7. Let f and g be real valued functions .
   (a) Show that $\min(f, g) = \frac{1}{2}|f + g| - \frac{1}{2}|f - g|$.
   (b) Show that $\min(f, g)$=-$\max(-f, -g)$.
   (c) Use (a) and (b) to prove that if f and g are continuous at x_0 in $\mathbb{R}$,then $\min(f, g)$ is continuous at x_0.

8. Prove that the following functions are discontinuous at the indicated points.
   (a) $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0, x_0 = 0$.
   (b) $g(x) = sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0, x_0 = 0$.

9. Prove each of the following functions in continuous at x_0 by verifying the $\epsilon - \delta$ definition.
   (a) $f(x) = x^2, x_0 = 2$.
   (b) $f(x) = \sqrt{x}, x_0 = 0$.
   (c) $f(x) = xsin(\frac{1}{x})$ for $x \neq o$ and $f(0) = 0, x_0 = 0$
   (d) $g(x) = x^3, x_0$ arbitrary.

10. (a) Let f be a continuous real valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$.
    (b) Let f and g be continuous real valued functions on (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b). Prove that $f(x) = g(x)$ for all $x \in (a, b)$.

11. State and prove Intermediate Value Theorem.

12. Let f be a continuous real valued function on a closed interval $[a, b]$. Then prove that f is a bounded function. Also prove that, f assumes its maximum and minimum values on $[a, b]$.

13. Let f be a continuous function mapping $[0, 1]$ into $[0, 1]$.Show that f has a fixed point.

14. Suppose f is a real valued continuous function on $\mathbb{R}$ and $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove there exist x between a and b such that $f(x) = 0$.

15. Prove that a polynomial function f of odd degree has at least one real root. Let $f(x) = sin(\frac{1}{x})$ for $x \neq 0$ and let $f(0) = 0$.
    (a) Show that f is discontinuous at 0.
    (b) Show that f has the intermediate value property on $\mathbb{R}$.

16. If f is continuous on a closed interval a, b then prove that f is uniformely continuous on $[a, b]$.

17. Show that the function $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$.

18. Which of the following continuous functions are uniformly continuous on specified set?Justify your answer.

    (a) $f(x) = x^3$ on $[0, 1]$.

    (b) $f(x) = x^2 sin(\frac{1}{x})$

    (c) $f(x) = tanx$ on $[0, \frac{\pi}{4}]$.

    (d) $f(x) = \frac{1}{x-3}$ on $(0, 3)$.

19. State and prove Chain Rule for differentiability.

20. If f is differentiable function at a, then prove that f is continuous at a.Is converse true? Justify.

21. Let f and g are differentiable functions at a. Then prove that $f + g, fg, cf$where c is arbitrary constant,$\frac{f}{g}$ for$g(a) =\neq 0$ are also differentiable at a. Also form the formulas.

22. State and prove Leibnitz Rule for differentiability.

23. let $h(x) = \sqrt{x}$ for$x \geq 0$. Use the definition of derivative to prove $h^{'}(x)\frac{1}{2}x^{-1/2}$ for $x > 0$.

24. Let $f(x) = x^2 sin(\frac{1}{x})$ for $x \neq 0, f(0) = 0$. Use the definition to show f is differentiable at $x = 0$ and $f^{'}(0) = 0$. Also prove that $f^{'}$ is not continuous at $x = 0$.

25. Let $f(x) = x^2$ for x rational and $f(x) = 0$ for x irrational.

    (a) Prove that f is continuous at $x = 0$.

    (b) Prove that f is discontinuous at $x \neq 0$.

    (c) Prove that f is differentiable at $x = 0$

26. State and prove Mean Value Theorem.

27. State and prove Rolle's Theorem.

28. Let f be a differentiable function on (a, b) such that $f^{'}(x) = 0$ for all $x \in (a, b)$. Then prove that f is constant on (a, b).

29. State and prove L'Hospital's Rule.

30. State and prove Generalized Mean Value Theorem.

31. State and prove Taylor's Theorem.

32. Find Taylor's series of the following function about $x = 0$.

    (a) e^x. —part $sinx$.

    (b) $cosx$.

    (c) $tanx$.

    (d) $log(x + 1)$.

(e) $\frac{1}{x+1}$.

(f) $\frac{1}{x-1}$.

33. State and prove Binomial Series Theorem.

34. Let f be a differentiable function on an interval (a, b). Then prove that f is strictly increasing if $f^{'}(x) > 0$ for all $x \in (a, b)$.

35. Let f be a differentiable function on an interval (a, b). Then prove that f is strictly decreasing if $f^{'}(x) < 0$ for all $x \in (a, b)$.

36. Let f be a differentiable function on an interval (a, b). Then prove that f is increasing if $f^{'}(x) \geq 0$ for all $x \in (a, b)$.

37. Let f be a differentiable function on an interval (a, b). Then prove that f is decreasing if $f^{'}(x) \leq 0$ for all $x \in (a, b)$.

38. Let f and g be differentiable on (a, b) such that $f^{'} = g^{'}$ on (a, b). Then prove that there exists a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$.

39. Let f be a one-to-one continuous function on an open interval I, and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and if $f^{'}(x_0) \neq 0$, then prove that f^{-1} is differentiable at $y_0 = f(x_0)$ and $(f^{-1})^{'}(y_0) = \frac{1}{f^{'}(x_0)}$.