ANEKANT EDUCATION SOCIETY'S

TULJARAM CHATURCHAND COLLEGE

OF ARTS, SCIENCE AND COMMERCE, BARAMATI

AUTONOMOUS

QUESTION BANK

FOR

M.Sc. Part- I (Sem-II)

STATISTICS

STAT 4201: Probability Theory

For ONE Mark each

Q. 1. Choose the correct alternative of the following

i) A field is a class closed under

a) complimentation and finite intersection

b) complimentation and finite union

c) finite union and finite intersection

d) both a) and b)

ii) The number of sets contained in largest field defined on set Ω={a, b, c, d} is

a) 10 b) 8

c) 16 d) 14

iii) A finite linear combination of indicators of set is a ….. function

a) elementary b) one-to-one

c) simple d) none of these

iv) For a sequence of sets $\{A_n, n \in N\}$, then which of the following relation between limit superior ($\overline{\lim} A_n$) and limit inferior ($\underline{\lim}\, A_n$) is true always?

a) $\underline{\lim}\, A_n \subset \overline{\lim} A_n$ b) $\overline{\lim} A_n \subset \underline{\lim}\, A_n$

c) $\underline{\lim}\, A_n = \overline{\lim} A_n$ d) $\overline{\lim}\, A_n \geq \lim A_n$

v) If $P_1(.)$ and $P_2(.)$ are probability measures on some measurable space, then $Q(.) = \alpha * P_1(.) + (1- \alpha) * P_2(.)$ is a probability measure, if

a) $\alpha \in R$ b) $0 \leq \alpha \leq 1$ c) $\alpha = 0.5$ d) $-1 \leq \alpha \leq 1$

vi) Characteristic function of random variable X is real, if

a) X is discrete random variable b) X is continuous random variable

c) X is symmetric about origin d) X is mixture random variable

vii) Probability measure is always

a) Non-negative b)Monotonic

c) Countably additive d) all of these

viii) The characteristic function $\varphi_X(t)$ of a r.v. X is ......

a) always real valued b) real, if X is symmetric around zero

c) always complex valued d) None of these

ix) A class closed under complementation and countable intersections is also closed under ......

a) countable union b) finite union

c) finite intersection d) all of these

x) Lebesgue measure of a singleton set is .....

a) Zero b) one c) cannot be determined d) none of these

Q. 2. State whether the following statements are TRUE or FALSE. **(1 each)**

i) Every σ-field contains φ and Ω.

ii) A sequence of sets always converges.

iii) Every σ-field is a field.

iv) The characteristic function uniquely determines the distribution.

v) A class containing only φ and Ω is a field.

vi) A mixture of two discrete random variables may be continuous random variable.

vii) Every field is a σ-field.

viii) The characteristic function of a real valued random variable always exists.

ix) Every subset of real line is a Borel set.

x) Every monotonic sequence of sets converges.

xi) Every σ-field is a field.

xii) Expectation of a random variable is always non-negative.

xiii)

- **<u>For TWO Marks each</u>**

Q. 1. Define the following terms with an illustration:

i. Minimal σ-algebra
ii. Point of Discontinuity
iii. Probability Measure
iv. Economical definition of random variable
v. Distribution Function
vi. Measurable function.
vii. Monotone class.
viii. Probability measure.
ix. Probability space.
x. Characteristic Function.
xi. Measurable Space.
xii. Field
xiii. Lebesgue-Stieltje's measure
xiv. Expectation of a discrete random variable.
xv. Monotonic sequence of sets.
xvi. Simple random variable.
xvii. σ-field
xviii. Independence of two events.
xix. Class of independent events.
xx. Strong Law of Large Number
xxi. Convergence of probability
xxii. Convergence in quadratic mean
xxiii. Independence of two classes of events
xxiv. π- system
xxv. Weak Law of Large Number
xxvi. Independence random variable
xxvii. Elementary random variable.
xxviii. Lebesgue – measure
xxix. Set of mutual convergence
xxx. Convergence in mean

Q. 2. Define characteristic function of a random variable.

Q. 3. State Parseval identity of characteristic function.

Q. 4. State inversion theorem of characteristic function.

Q. 5. Explain Borel σ-field.

Q. 6. Define and illustrate Lebesgue-Stieltje's measure.

Q. 7. Give an example for the following cases

i. A Class of sets which is a field but not a σ- filed.
ii. A Class of sets which is a field as well as a σ- filed.
iii. lim sup An = lim inf An.
iv. lim sup An ≠ lim inf An.
v.

Q. 8. State the following theorems.

i) Borel 0-1 law
ii) Lévy continuity theorem
iii) Liapounov's form of central limit theorem
iv) Fatou's theorem

Q. 9. Prove the following results

i. $(\limsup A_n)^c = \underline{\lim}\, A_n^c$
ii. Every continuous function is a Borel function.
iii. Intersection of two σ -field is also σ -field.
iv. Probability measure is σ-additive.
v. Distribution function is a non-decreasing function.
vi. $(\limsup A_n)^c = \underline{\lim}\, A_n^c$
vii. Every continuous function is a Borel function.
viii. Intersection of two σ -field is also σ -field.
ix. $P(AB^c \cup BA^c) = P(A) + P(B) - 2P(AB)$
x. Probability measure is σ-additive.

<u>For Three Marks each</u>

Q. 1. Prove or disprove the following

i. If **A** is a field then it is closed under intersection and hence closed under difference of sets.
ii. Let **A =** {A ⊂ Ω: Either A is finite or A^c is finite} then **A** is an algebra.
iii. Arbitrary intersection of fields is a field.
iv. Arbitrary intersection of σ- fields is a σ- field.
v. Distribution function is a left continuous function.

vi. Distribution function is a right continuous function.

vii. Inverse image of a σ-field is always a σ-field.

viii. $\overline{\lim}\,(A_n \cup B_n) = \overline{\lim}\,A_n \cup \overline{\lim}\,B_n$

ix. $\overline{\lim}\,(A_n \cap B_n) = \overline{\lim}\,A_n \cap \overline{\lim}\,B_n$

x. $\underline{\lim}\,(A_n \cup B_n) = \underline{\lim}\,A_n \cup \underline{\lim}\,B_n$

xi. $\underline{\lim}\,(A_n \cap B_n) = \underline{\lim}\,A_n \cap \underline{\lim}\,B_n$

xii.

Q. 2. Find characteristic function of the following random variables X whose distribution is

i. N (μ, σ^2)

ii. C(μ, λ)

iii. P(λ)

iv. Bin(n, p)

v. Double Exp (μ, λ)

Q. 3. If $0 \le X_n \uparrow X$, then show that $E(X_n) \uparrow E(X)$.

Q. 4. If A_n = A, if n = 1, 3, 5, …. and A_n = B, if n = 2, 4, 6, … Then obtain $\underline{\lim}\,A_n$, $\overline{\lim}\,A_n$ When does lim A_n exist?

Q. 5. In usual notation show that $P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B)$.

Q. 6. If $A_n \to A$ then show that $A_n^c \to A^c$.

For Four Marks each

Q. 1. State and prove disjointification lemma.

Q. 2. State the two distributive properties of sets. Prove any one distributive property using indicator functions and deduce the other.

Q. 3. Define 'limit inferior' and 'limit superior' of a sequence of sets in terms of all but a finite and infinitely many and also in terms of unions and intersections.

Q. 4. Give examples of sequences $\{A_n\}$ and $\{B_n\}$ such that both the sequences do not converge, but $\{A_n \cap B_n\}$ and $\{A_n \cup B_n\}$ both converge.

Q. 5. Determine whether the sequences $\{A_n\}$, $\{B_n\}$, $\{A_n \cap B_n\}$ and $\{A_n \cup B_n\}$ converge

$A_{2n-1} = \{1, 2, \cdots, 2n-1\}$ and $A_{2n} = \{2n\}$;
$B_{2n-1} = \{2n-1\}$ and $B_{2n} = \{1, 2, \cdots, 2n\}$.

Q. 6. Give an example of a convergent sequence of sets that is not monotone.

Q. 7. Show that a σ-field is a field. Is the converse true? Justify.

Q. 8.

Q. 9. Find lim inf A_n and lim sup A_n for the following sequence $\{A_n\}$,

i. $A_n = \left(2 - \frac{1}{n+1}, 4\right),\ n \in N$

ii. $A_n = \left(2 - \frac{(-1)^n}{n+1},\ 3\right),\ n = 1, 2, 3, \ldots$

iii. $A_n = \left(a - \frac{1}{n},\ b + \frac{1}{n}\right)$

iv. $A_n = \left[a - \frac{1}{n},\ b + \frac{1}{n}\right]$

v. $A_n = \left(a - \frac{1}{n},\ b + \frac{1}{n}\right]$

vi. $A_n = \left[a - \frac{1}{n},\ b + \frac{1}{n}\right)$

vii. $A_n = \left(a + \frac{1}{n},\ b - \frac{1}{n}\right)$

viii. $A_n = \left[a + \frac{1}{n},\ b - \frac{1}{n}\right]$

ix. $A_n = \left(a + \frac{1}{n},\ b - \frac{1}{n}\right]$

x. $A_n = \left[a + \frac{1}{n},\ b - \frac{1}{n}\right)$

xi. $A_n = \left(a + \frac{1}{n},\ b + \frac{1}{n}\right)$

xii. $A_n = \left[a + \frac{1}{n},\ b + \frac{1}{n}\right]$

xiii. $A_n = \left(a + \frac{1}{n},\ b + \frac{1}{n}\right]$

xiv. $A_n = \left[a - \frac{1}{n},\ b + \frac{1}{n}\right)$

xv. $A_n = \left(a - \frac{1}{n},\ b - \frac{1}{n}\right)$

xvi. $A_n = \left[a - \frac{1}{n},\ b - \frac{1}{n}\right]$

xvii. $A_n = \left(a - \frac{1}{n},\ b - \frac{1}{n}\right]$

xviii. $A_n = \left[a - \frac{1}{n},\ b - \frac{1}{n}\right)$

xix. $A_n = \{n, n + 1, \cdots\}$.

xx. $A_n = \{1, 2, \cdots, n\}$.

xxi. $A_n = (-\infty, r_n)$, *where* $r_n \downarrow x$ *as* $n \to \infty$.

xxii. $A_n = (-\infty, r_n]$, *where* $r_n \uparrow x$ *as* $n \to \infty$.

xxiii.

xxiv.

xxv.

Q. 10. Prove that P(lim inf A_n) ≤ lim inf P (A_n) ≤ lim sup P(A_n) ≤P(lim sup A_n).

Q. 11. If $\{A_n\}$ is a increasing sequence of sets in A then show that P(lim A_n)=lim P (A_n).

Q. 12. If $\{A_n\}$ is a monotonic sequence of sets which decreases to A. Then prove that P(A_n) also decreases to P(A).

Q. 13. If $A_n \to A$ then show that $P(A_n) \to P(A)$.

Q. 14. Define bivariate distribution function and state all its properties.

Q. 15. Write a short note on Lebesgue measure.

Q. 16. Suppose $X_1, X_2, \ldots, X_n$ be a r.s. from U (0, θ). Define $X_{(n)} = \text{Max}\{X_1, X_2, \ldots, X_n\}$ then show that $X_{(n)} \xrightarrow{P} \theta$.

Q. 17. If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ then show that $X_n Y_n \xrightarrow{P} XY$.

Q. 18. If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ then show that $X_n + Y_n \xrightarrow{P} X+Y$

Q. 19. If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ then show that $aX_n + bY_n \xrightarrow{P} aX+bY$

Q. 20. If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ then show that $\frac{X_n}{Y_n} \xrightarrow{P} \frac{X}{Y}$ provided $Y_n \neq 0\ \forall\ n \geq 1$ and $Y \neq 0$.

Q. 21. If $X_n \xrightarrow{D} X$ *and* $Y_n \xrightarrow{P} c$ then show that $X_n Y_n \xrightarrow{D} cX$.

Q. 22. If $X_n \xrightarrow{r} X$ then show that $X_n \xrightarrow{p} X$. Is converse true? Justify your answer.

Q. 23. Show that if $X_n \xrightarrow{P} X$ then $X_n \xrightarrow{L} X$.

Q. 24. State and prove invariance property of convergence in probability under continuous mapping.

Q. 25. In usual notation show that, $E|X_n - X|^r \to 0$ implies $E|X_n|^r \to E|X|^r$.

Q. 26. State and prove Borel-Cantelli lemma.

Q. 27. State and prove any two properties of sigma algebra.

Q. 28. Sate and prove any two properties of Probability measure.

For FIVE Marks each

Q. 1. If $|X_n| \le Y$, Y integrable and $X_n \xrightarrow{a.s.} X$, then prove that $E(X_n) \to E(X)$.

Q. 2. State and prove Hölder's inequality.

Q. 3. Show that: $I_{\limsup A_n} = \limsup I_{A_n}$. $I_{\liminf A_n} = \liminf I_{A_n}$

Q. 4. Show that: $\limsup A_n^C = (\liminf A_n)^C$

Q. 5. Show that : $\liminf A_n^C = (\limsup A_n)^C$

Q. 6. Show that, a decreasing sequence of sets converges and identify the limit. Further, state and deduce the corresponding result for an increasing sequence of sets.

Q. 7.

Q. 8. Show that a Borel function of a random variable X is a random variable.

Q. 9. Define inverse image of a set. Show that inverse image of a σ-field is also a σ-field.

Q. 10. In usual notation show that, $\sigma\left(X^{-1}(\varepsilon)\right) = X^{-1}\left(\sigma(\varepsilon)\right)$

Q. 11. State and prove necessary and sufficient condition for convergence in probability.

Q. 12. If X and Y are arbitrary random variables, then show that $E(XY) = E(X) \cdot E(Y)$

Q. 13. Prove that if X and Y are independent random variables and g(.) and h(.) are two Borel functions of random variables X and Y respectively, then g(X) and h(Y) are independent random variables.

Q. 14. If $E|X|^r < \infty$, then show that $E|X|^k < \infty$ for $0 < k \le r$ and $E X^k$ exist and is finite for $k < r$, where k is an integer.

Q. 15. State and prove Weak Low of Large Number.

Q. 16. State and prove Central Limit Theorem.

Q. 17. Show that every function X on Ω is measurable with respect to power set of Ω.

Q. 18. Prove that for A, B and C arbitrary, P ($(A \Delta B) \le P(A \Delta C) + P(B \Delta C)$ When is equality attained?

For SIX Marks each

Q. 1. Let Ω = { x_1, x_2, x_3, x_4 }, P{ x_1 } = 1/6, P{ x_2 } =1/3, P{ x_3 } = 1/5, P{ x_4 } =3/10. Define sequence of sets {A_n; n ≥1} such that, A_n = {x_1, x_2} if n is even and A_n = {x_2, x_3} if n is odd. Find P(lim sup A_n), P(lim inf A_n), lim inf P (A_n) and lim sup P(A_n) also verify the relation.

Q. 2. Show that a finite field is a σ-field and give an example of a field which is not a σ-field.

Q. 3. Prove that a monotone field is a σ-field.

Q. 4. Show that a class which is both a π-class and λ-class is a σ-field.

Q. 5. Show that a λ-calss is a monotone class.

Q. 6. Show that intersection of σ-fields is a σ-field, but union is not.

Q. 7. Define the σ-field generated by the class $\mathcal{C}$ and show that $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_2)$, if $\mathcal{C}_1 \subseteq \mathcal{C}_2$.

Q. 8.

Q. 9. Show that, lim inf $A_n \subseteq$ lim sup A_n. Give an example of a sequence of sets for which strict inequality holds in and one example for which equality holds.

Q. 10. Suppose $\underline{X}$ = (X_1, X_2) : (**Ω**,**A**) → (**R** $_2$, **B** $_2$); X is a random vector w. r. t. **A** if and only if X_1 and X_2 are random variable w. r. t **A**.

Q. 11. If {X_n} is a sequence in r. v. then show that lim sup X_n, lim inf X_n is also a r.v.

Q. 12. Define minimal σ–field. Show that inverse image X^{-1} of a minimal σ-field over any class C is minimal σ –field over $X^{-1}(C)$.

Q. 13. Define σ-field and monotone class. Prove that every σ-field is a monotone class. Is the converse true? Justify your answer.

Q. 14. Define distribution function of a random variable. If X is a continuous random variable with distribution function F, then show that $E(X) = \int_0^\infty (1 - F(x))dx$.

Q. 15. Define expectation of a random variable. Show that for a random variable X, E(X) exists if and only if E(|X|) exists.

Q. 16. State and prove Cauchy-Schwartz inequality.

Q. 17. State and prove Lyapunov's inequality.

Q. 18. State and prove Jensen's inequality.

Q. 19. State and prove C_r inequality.

Q. 20. Define characteristic function of a random variable. Prove inversion theorem of this function.

Q. 21. Sate and prove any two properties of Probability measure.

Q. 22. Prove that probability function is a continuous from above function as well continuous from below.

Q. 23. State and prove continuity property of probability measure.

Q. 24. State and prove uniquencess theorem of characteristics function.

Q. 25. Define a random variable. If X and Y are two random variables, then prove that E(a X + b Y) = a E(X) + b E(Y).

Q. 26. State and prove monotone convergence theorem.

Q. 27. State and prove Basic inequality.

Q. 28. If A_1 and A_2 are measurable sets and a function X is defined by

$$X(w) = \begin{cases} -1, & w \in A_1 \\ +1 & w \in A'_1 A_2 \\ 0 & w \in A'_1 A'_2 \end{cases}$$

Examine whether X is measurable.

Q. 29. Let $\Omega = \{-2, -1, 3, 7\},\ A = \{-2, 3, 7\}, \mu(-2) = -2,\ \mu(-1) = 1,\ \mu(3) = 3,\ \mu(7) = 7.$

If $X(\omega) = \begin{cases} 1, & if\, \omega = 3,\ \omega = 7 \\ -1, & if\, \omega = -2,\ \omega = -1 \end{cases}$

Evaluate $\int_A X \, d\mu$.

Q. 30. Define expectation of a simple random variable. Prove any three properties of it.

Q. 31. State and prove multiplication theorem for two random variables.

Q. 32. Define expectation of a non- negative random variable. Prove any three properties of it.

Q. 33. Let $\Omega = \{a, b, c, d\}$, A $= \{\phi, \Omega, \{a, b\}, \{c, d\}\}$, $X(a) = X(b) = -1,\ X(c) = 1, X(d) = 2.$

Examine whether *X* is A-measurable. Give an example of a function *Y*, different than *X*, which A measureable.

Q. 34. Define positive part and negative part of a random variable. Also prove that X is integrable if and only if |X| is integrable.

Q. 35. State and prove Kolmogorov 0-1 law.

Q. 36. If X_n is a sequence of random variable then prove that $X_n \xrightarrow{P} 0$ if and only if $E\left[\frac{|X_n|}{1+|X_n|}\right] \xrightarrow{p} 0$ as n $\to \infty$.

Q. 37. State and prove necessary and sufficient condition for convergence in probability.

Q. 38. If $\{X_n\}$ converges in probability, then show that it is Cauchy in probability.

Q. 39. Define Characteristic function φ (.). Show that φ is continuous.

Q. 40. Find the distribution function if character function of X is is $\frac{1}{4}(1 + e^{iu})^2$

Q. 41. A sequence of random variables converges almost surely to a random variable iff the sequence converge mutually almost sure.

Q. 42. If $\{X_n\}$ is sequence of independent identically distributed random variables with $E(X_i) = \mu < \infty$ for all $i \geq 1$ then the sequence $\bar{X}_n \xrightarrow{P} E(X_i) = \mu$ where $\bar{X}_n = \frac{\sum Xi}{n}$.

Q. 43. State and prove Khinchine's Weak Law of Large Number.

Q. 44. Joint probability density function of (X_1, X_2, X_3) is

$$f(x_1, x_2, x_3) = \begin{cases} 1/2 + 4x_1x_2x_3 & , 0 \leq x_1, x_2, x_3 \leq 1, \\ 0 & , otherwise \end{cases}$$

Examine whether X_1, X_2, X_3 are

i) Pairwise independent

ii) Mutually independent

<u>For EIGHT Marks each</u>

Q. 1. Define limit superior and limit inferior of a sequence of sets. Hence find the same for sequence $\{A_n\}$, where

i. $A_n = \left(2 - \frac{(-1)^n}{n}, 5 + \frac{(-1)^n}{n}\right), \quad n \in N$

ii. $A_n = \left(0, \frac{(-1)^n}{2n}\right), \quad n \in N.$

Q. 2. Define Borel σ-field. Obtain the types of set which are members of Borel σ-field. Justify your answer.

Q. 3. If sequence of seats $\{A_n\}$ is $A_{2n} = \left(0, \frac{1}{2n}\right)$ and $A_{2n+1} = [-1, \frac{1}{2n+1}]$ then examine whether sequence of set $\{A_n\}$ is convergent, if convergent derive the limits.

Q. 4. Let $F(x)$, $x \geq 0$ be a distribution function.

Define $G(x) = \begin{cases} 0 & x \leq 0 \\ \exp\left[1 - \alpha\left(1 - F(x)\right)\right], & x > 0, \alpha > 0 \end{cases}$

Is $G(x)$ a distribution function? Justify your answer. Is $G(x)$ a continuous function? Explain.

Q. 5. Define inverse mapping. Prove that inverse mapping preserves all the set relations.

Q. 6. Define characteristic function of a random variable X. Prove any three properties of characteristic function.

Q. 7. If $X_1, X_2, \ldots, X_n$ are independent random variables, then show that the characteristic function of $X_1 + X_2 + \ldots + X_n$ is the product of the characteristic function of X_k's.

Q. 8. Prove that any arbitrary random variable can be expressed as a limit of sequence of simple random variables.

Q. 9. Define expectation of an arbitrary random variable. Prove any three properties of it.

Q. 10. If X and Y are bivariate random variable with

F (x, y) =	0	if $x < 0$, $y < 0$ or $x + y < 1$
	1	else where

Compute $P[\frac{1}{2} < X \leq 1, \frac{1}{2} < Y \leq 1]$. Hence Comment on F (x, y)

Q. 11. Obtain the mean and variance of the following distribution

$$F(x) = \begin{cases} 0 & x < 0 \\ p + (1-p)(1 - e^{-\lambda x}) & 0 < x < T \\ 1 & x \geq T \end{cases}$$

Q. 12. Show that a Borel function of A-measurable function of X is A-measurable. What is relation between σ-field induced by X and σ-field induced by Borel function of X?

Q. 13. Define expectation of a simple random variable. If X and Y are two simple random variables, then prove that $E(X + Y) = E(X) + E(Y)$ and $E(cX) = c\,E(X)$, where c is a real number.

Q. 14. State and prove Fatou's Lemma.

Q. 15. Determine whether weak law of large number holds for the sequence of independent random variables with *pmf* $P[X_n = 2^n] = \frac{1}{2} = P[X_n = -2^n]$; n = 1, 2, …

Q. 16. Show that sub-classes of independent classes are independent. Hence prove that if X and Y are independent random variables and g(.) and h(.) are two Borel functions of random variables X and Y respectively, then g(X) and h(Y) are independent random variables.

Q. 17. Suppose $\{X_n; n \geq 1\}$ be a sequence of i.i.d random variables with common mean μ and common variance $\sigma^2 \in (0, \infty)$. Let $S_n = \sum X_i$ prove that in the long run E (S_n) behaves like median of S_n.

Q. 18. Find the variance of the rectangular distribution on (a, b). Hence obtain a lower bound to the probability of the interval $\left[\left(\frac{3a-b}{2}\right), \left(\frac{3b-a}{2}\right)\right]$.

Q. 19. Suppose X is logistic r.v. with distribution function $F(x) = \frac{1}{(1-e^{-(ax+b)})}, a > 0, x \in$ R. Show that pdf f and df F are related by $f(x)$ = a $F(x)$ (1 - $F(x)$).

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