Anekant Education Society's

Tuljaram Chaturchand College, Baramati

Department of Mathematics

Class: M.Sc. -ISubject: Linear Algebra

Question Bank

Short questions

1) Let V=R^n and W be a subset of V consisting of vectors $[x_1 \ldots \ldots . x_n]^t$ such that $x_n = 0$,

Is W subspace of V?Justify.

2) Let V=R^2 and W be a subset of V consisting of vectors $[x_1 \ldots \ldots . x_n]^t$ such that

$x_1 + x_2 + \ \ldots \ldots . +x_n = 0$,Is W subspace of V?Justify.

3) Consider the vector space K^{nXn} over K . Then prove that set of all diagonal matrices are

subspace of K^{nXn}.

4) Consider the vector space K^{nXn} over K . Then prove that set of all upper triangular

matrices are subspace of K^{nXn}.

5) Define Linear Transformation.

6) Let V and V' be finite dimensional vector space over K of dimension n and m resp.then find the dimension of L(V,V').

7) Whether the map $T: R^2 \rightarrow R^3$ defined as $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ is linear?

8) Whether the map $T: R^2 \rightarrow R^3$ defined as $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ y \end{bmatrix}$ is linear?

9) Whether the map $T: R^2 \rightarrow R^3$ defined as $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+1 \\ y \\ z \end{bmatrix}$ is linear?

10) What is idempotent operator?

11) Give an two examples of idempotent operator.

12) Write First Isomorphism Theorem.

13) Write Second Isomorphism Theorem.

14) Give the dimension formula for addition of two subspaces.

15) Give change of basis formula.

16) Define T-invariant subspaces.

17) Give two examples of T-invariant subspaces.

18) Define annihilator of Subset S of vector space V.

19) Let V and W be a finite dimensional vector space over K, and let T$\in L(V,W)$. Then prove that rank T = rankT^*.

20) State Eigen value and Eigen vector of linear operator.

21) Find characteristic polynomial of A=$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

22) Define similar matrices.

23) Give any two properties of similar matrices.

24) What is relation between algebraic multiplicity and geometric multiplicity of Eigenvalue?

25) State true or false with justification, Eigenvectors corresponding to distinct eigenvalues are linearly independent.

26) State true or false with justification, a nonzero nilpotent operator on V is diagonalizable.

27) Explain Jordan chain.

28) Write possible Jordan canonical form if characteristic polynomial is (x-2)$(x-9)^2$.

29) Define Inner product function.

30) Write parallelogram law.

31) Write the statement of Spectral theorem.

32) Explain the matrixform of a bilinear form.

Long Questions

33) Let V be a vector space over K. Show that for $\alpha \in K$ and x$\in V, \alpha x = x$ then $\alpha = 1\ or\ x = 0$.

34) Let $V = K^2$.Define coordinate wise addition on V, and scalar multiplication:

$\alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ 0 \end{bmatrix}; \alpha \in K, \begin{bmatrix} x \\ y \end{bmatrix} \in K^2.$

Is V a vector space with respect to these operations?

35) Let V=R^n and W be a subset of V consisting of vectors $[x_1 \dots \dots . x_n]^t$ such that

i)$x_1 + x_2 + \dots \dots . + x_n \geq 0$, ii) $x_1 + x_2 + \dots \dots . + x_n = 1$.

In which of above conditions, W subspace of V?

36) Consider the vector space R[X] over which of the following subsets of R[x] are subspace:

i) $\{p(x)|\ p(x) = p(1-x) for\ all\ x\}$, ii) $\{p(x)|\ p(1) \geq 0\}$.

37) Let a=$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$,b = $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, c = $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and d= $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ be a vectors inR^3.

Let $W_1 = \langle a, b\rangle, W_2 = \langle b, c\rangle, W_3 = \langle c, d\rangle$.Show that$W_1 = W_3$,$W_1 \cap W_2 = \langle b\rangle$.

Also identify the subspace $W_1 + W_2 . Is\ W_1 \cup W_2$ a subspace of R^3 ?

38) Let X be a subset of a vector space V. Show that$\langle X\rangle = \langle\langle X\rangle\rangle$.

39) Let X and Y be subsets of a vector space. Show that$\langle X \cup Y\rangle = \langle X\rangle + \langle Y\rangle$.

40) Let V be a vector space over K. Then prove that a finite subset B of V is a basis of V if and Only if every element of V is a unique linear combination of elements of B.

41) Let V be a finite dimensional vector space over K, and let X and Y be finite subsets of V. If Y is linearly independent and V = $\langle X\rangle$,then prove that $|Y| \leq |X|$.

42) Prove that a linearly independent subset of a finite dimensional vector space can be extended to form a basis of the vector space.

43) Let K be a finite field with q elements, and let V be vector space over K of dimension n. Then find number of distinct ordered bases of V.

44) Let K be a finite field with q elements, and let V be vector space over K of dimension n.

Then find number of distinct unordered bases of V.

45) Let V be a subspace of R^5 generated by $v_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 7 \\ -5 \\ 6 \\ 5 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ 6 \\ -3 \\ 0 \\ 13 \end{bmatrix}$, and let W be a

Subspace generated by $w_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 5 \\ 16 \\ -3 \\ 12 \\ 6 \end{bmatrix}$, $w_3 = \begin{bmatrix} 3 \\ 8 \\ 3 \\ 4 \\ 2 \end{bmatrix}$.

Then find a basis for $V + W$ and $V \cap W$.

46) Prove that a matrix A ∈ $K^{m\times n}$ is invertible if and only if the columns of A form a basis of K^n.

47) Find a two dimensional subspace of R^4 which does not contain vectors $[1\ 3\ 2\ 5]^t$ and $[2\ 4\ 3\ 1]^t$.

48) Let V and V' be vector space over K , and let T:V→ V' be a linear transformation then prove the followings:

$i)\ T(0) = 0, ii)\ T(-v) = -T(v), v \in V, iii)\ T(U)$ is a subspace of V' whenever U is a subspace of V, iv) $T^{-1}(U')$ is a subspace of V , whenever U' is a subspace of V'.

49) Let T:V→ V' be a linear transformation then prove T is injective if and only if Ker T={0}.

50) Let V and V' be finite dimensional vector spaces over K of dimensions n and m respectively.

Then prove that dimL(V, V') = nm.

51) Let V and V' be finite dimensional vector spaces over K .Then prove that $V \cong V'$ if and only if dim V=dim V'.

52) Verify that T:$R_n[x] \to R^{(n+1)\times(n+1)}$, n≥ 1,given by

T($a_0 + a_1x + \ldots + a_nx^n$) $= \begin{bmatrix} a_0 & a_1 & \ldots & a_n \\ 0 & \ddots & & \vdots \\ 0 & 0 & & a_0 \end{bmatrix}$ is an injective linear transformation.

53) Let T: $R^3 \to R^3$, T$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+2y-z \\ 2x+5y-2z \\ 4x+4y-2z \end{bmatrix}$.Verify that T is a linear operator on R^3.

Find the kernel and image of T.

54) Let S and T be idempotent on V. Prove that i) $I - T$ is an idempotent and ii) $S + T$ is an idempotent if and only if $ST = 0 = TS$.

55) State and prove Second isomorphism theorem.

56) State and prove Third isomorphism theorem.

57) Let W=$\langle [1\ 2\ 1\ 0\ 1]^t, [1\ 0\ 1\ 1\ 1]^t, [1\ 2\ 1\ 3\ 1]^t \rangle$, a subspace of R^5.Find a basis of $R^5/_W$.

58) Consider a linear operator T on R^3 defined as $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ 0 \\ z \end{bmatrix}$.Determine an isomorphism

S:$\frac{R^3}{Ker\ T} \to Im\ T$.

59) Let V_1, V_2... V_m be vector spaces over a field K. Prove that V = $V_1 \oplus ... \oplus V_m$ is finite Dimensional if and only if each V_i is finite dimensional. Also prove that dim$V_1 \oplus ... \oplus V_m$ = dimV_1 + ...+ dim V_m.

60) Let I = (– a, a), a > 0 be an open interval in R and let V = R^I , the space of all real valued functions defined on I. Show that V = $V_e \oplus V_0$, where V_e is the set of all even functions on I and V_0 is the set of all odd functions on I.

61) Consider the vector space R^3 [x] of polynomials with real coefficients and of degree at most 3. The differential operator D is a linear operator on R^3 [x]. Write the matrix representation of D with respect to $B_1 = \{1 + x, x + x^2, x^2 + x^3, x + x^3\}$.

62) Let T be a linear operator on R^3, $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+2y+2z \\ 2y+z \\ 2x+3y+z \end{bmatrix}$.Write the matrix representation of

TWith respect to basis $B = \{[1\ 1\ 0]^t, [-1\ 1\ 0]^t, [1\ 1\ 1]^t\}$.If B_0 is the standard basis of R^3,

Find an invertible matrix P such that $[T]_{B_0} = P^{-1}[T]_B P$.

63) Let V be a finite dimensional vector space over K, and let X and Y be subspaces of V.

Then prove the followings:

i) $(X+Y)^0 = X^0 \cap Y^0$,ii) $(X \cap Y)^0 = X^0 + Y^0$, iii) Under the canonical identification of $V^{**}$with V, $(X^0)^0$ = X.

64) Let V and W be finite dimensional vector spaces over K, and let $T \in L(V,W)$.Then prove That Ker$T^* = (imT)^0$ and imT^* = (Ker$T)^0$.

65) Find Eigen value and Eigen vector of matrix A=$\begin{bmatrix} 4 & 2 & 2 \\ 3 & 3 & 2 \\ -3 & -1 & 0 \end{bmatrix}$.

66) Find Eigen value and Eigen vector of matrix A=$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

67) Let V is a finite dimensional vector space over K and $T \in L(V)$, Prove that T is invertible if and Only if 0 is not an eigen value of T.

68) Let V is a finite dimensional vector space over K and $T \in L(V)$, v and w be eigenvectors of TCorresponding to two distinct eigenvalues $\lambda\ and\ \mu$ respectively then show that for a NonzeroScalars $\alpha\ and\ \beta, \alpha u + \beta w$ is not an eigenvector of T.

69) Let $A \in K^{n \times n}$, Show that A and A^t have the same characteristic polynomials.

70) Let V is a finite dimensional vector space over K and $let\ T$ be a linear operator on V .Let

Minimal polynomial of T in $K[x]$be $m_T(x) = P_1(x)^{r_1} \ldots \ldots \ldots \ldots P_k(x)^{r_k}$,where $P_1(x)$,$P_2(x)$,.......,$P_k(x)$ are monic irreducible polynomials and $r_1, r_2, \ldots, r_k$ are positive integers. ThenProve that V=$kerp_1(T)^{r_1} \oplus \ldots \ldots \ldots \ldots . \oplus kerp_k(T)^{r_k}$,a direct sum of T-invariant subspace of V.

71) Let V is a finite dimensional vector space over K and $let\ T$ be a linear operator on V.

If for each i ,T_i is linear operator on Ker $P_i T^{r_i}$ induced by T ,then prove that minimal

Polynomial of T_i is $p_i(x)^{r_i}$.

72) State and prove Cayley Hamilton Theorem.

73) Obtain minimal polynomial of A = $\begin{bmatrix} 3 & -2 & 2 \\ 10 & -9 & 10 \\ 6 & -6 & 7 \end{bmatrix}$.

74) Obtain minimal polynomial of B = $\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.

75) Give an example of a linear operator on R^4 whose minimal polynomial is

$(x^2-1)(x+1)$ *and characteristic polynomial is* $(x^2-1)^2(x+1)$.

76) Compute A^{25} , if A = $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix}$.

77) Prove that geometric multiplicity of an eigenvalue of a linear operator cannot exceed its

Algebraicmultiplicity.

78) Let V be a finite dimensional vector space over K of dimension n and let T be a linear

Operator on V. Then the following statements are equivalent:

i) the characteristic polynomial of T splits over K, ii) every nonzero T-invariant subspace

of V contains an eigenvector of T, iii) T is triangulable.

79) Let matrix A= $\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & -2 & -2 \\ 1 & 0 & -2 \\ 3 & -3 & -1 \end{bmatrix}$ then prove that

There exists an invertible matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.

80) Verify that 1 is eigenvalue of A and find its geometric and algebraic multiplicities

Where A= $\begin{bmatrix} 3 & 0 & 0 & -1 \\ -2 & 1 & 1 & 0 \\ 2 & 1 & 1 & -1 \\ 6 & 0 & -1 & 1 \end{bmatrix}$.

81) Give an example of a 3×3 matrix which is not triangulable over R but is

Diagonalizable over C. Explain it.

82) Let T be a linear operator on an n-dimensional vector space V over field K. If the Characteristic polynomial of T splits over K, then prove V has a basis which is a Disjoint union of Jordan chains for T, that is, V has a Jordan basis for T.

83) Determine the Jordan canonical form of matrix A= $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

84) Write all possible Jordan canonical forms if the characteristic polynomial is $(x-2)^4(x-3)^2$.

85) Determine the rational canonical form of a matrix whose characteristic polynomial Is $(x-1)^3(x-2)^2$ and minimal polynomial is $(x-1)^2$(x-2).

86) Let V be an inner product space over F and let u,v $\in V$.Then prove the followings:

i) $\|u \pm v\|^2 = \|u\|^2 \pm 2Re\,(u,v) + \|v\|^2$, where Rez is real part of complex number Z.

ii) $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$.

iii) $\|\lambda u\| = |\lambda|\|u\|$for all $\lambda \epsilon F$.

iv) 4(u,v) = $\begin{Bmatrix} \|u+v\|^2 - \|u-v\|^2 If\ F = R \\ \|u+v\|^2 - \|u-v\|^2 + i\|u+iv\| - i\|u-iv\|\ if\ F = C \end{Bmatrix}$

v) $|(u,v)| \leq \|u\|\|v\|$

vi) $\|u \pm v\| \leq \|u\| + \|v\|$.

vii) $|\|u\| - \|v\|| \leq \|u - v\|$.

87) Decide which of the following functions define inner product on R^2 .

For $x = [x_1, x_2]^t$, $y = [y_1, y_2]^t$;

i) (x,y) = $x_1y_2 + x_2y_1$,ii)(x,y) = $x_1y_2 - x_2y_1$,iii) (x,y) = $2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$.

88) Decide which of the following functions define inner product on C^2 .

For $x = [x_1, x_2]^t$, $y = [y_1, y_2]^t$;

i) (x,y) = $x_1\bar{y}_2$, ii) (x,y) = $x_1\bar{y}_1, + x_2\bar{y}_2$,iii) (x,y) = $2x_1\bar{y}_1 + i(x_2\bar{y}_1 - x_1\bar{y}_2) + 2x_2\bar{y}_2$.

89)Prove that similar matrices have the same characteristic polynomial but the converse is not true.

90) Let T be a self adjoint operator on a finite dimensional inner product space V. Prove that T is positive definite if and only if all eigenvalues of T are positive. Hence deduce that if T is a positive definite operator, then so is $T^{-1.}$

91) Prove that the eigenvalues of a self adjoint operator are real. Also prove that a normal operator with all real eigenvalues is self adjoint.

92) Consider standard inner product space R^3 with basis $\left\{x_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, x_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}\right\}$

Then using Gram-Schmidt orthonormalization find orthonormal basis.

93) Consider the inner product space $R_3[x]$ with the inner product(p(x), q(x)) $=\int_{-1}^{1} p(x)q(x)$dx.

Find the adjoint of the differential operator D.

94) Let W_1 and W_2 be subspaces of V. Prove that,

i) If $W_1 \perp W_2$, then $W_1 \cap W_2 = \{0\}$, ii) $W_1 \subseteq W_2$ if and only if $W_2{}^{\perp} \subseteq W_1{}^{\perp}$.

95) Prove that a bilinear form is reflexive if it is either symmetric or alternating.

96) State and prove Riesz representation theorem.

97) Decide which of the following mappings $\phi: R^2 \times R^2 \to R$ are bilinear,ifFor

$x = [x_1 , x_2]^t, y = [y_1 , y_2]^t$ in R^2 :

i) $\emptyset(x, y) = x_1y_1 + 1$,ii) $\emptyset(x, y) = x_1x_2 + y_1y_2 - x_2y_1 - x_1y_2$.

98) Find polar decomposition of $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

99) State and prove spectral theorem.

100) Find a unitary matrix whose first two columns are ,

$\left[\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} , \frac{1}{\sqrt{3}}\right]^t$, $\left[0, \frac{1}{\sqrt{2}} , \frac{1}{\sqrt{2}}\right]^t$.