Anekant Education Society's

Tuljaram Chaturchand College of

Arts, Commerce and Science, Baramati

(Autonomous)

QUESTION BANK

FOR

M.Sc SEM-I

STATISTICS

PAPER: STAT-4102

Linear Algebra -4 Credit

(With effect from June 2019)

Unit 1:A) Define the following terms with one illustration

1. Matrix
2. Square Matrix
3. Row Matrix
4. Column Matrix
5. Null Matrix
6. Diagonal Matrix
7. Scalar Matrix
8. Identity Matrix
9. Symmetric Matrix
10. Skew Symmetric Matrix
11. Upper traingular / Lower traingular matrix
12. Similar Matrix
13. Horizontal Matrix./Vertical Matrix
14. Unitary Matrix.
12. Idempotent Matrix
13. Nilpotent Matrix
14. Orthogonal Matrix.
15.Elementary Matrix.

B) Choose the correct alternative of the following:

1. The rank of $A=\begin{bmatrix} 2 & 3 & 0 \\ 3 & 3 & 6 \\ 2 & 2 & 4 \end{bmatrix}$ is

a) 1 b) 2 c) 3 d) none of these.

2. Rank of every non -singular matrix of order n is

a) n+1 b) n c) n-1 d) none of these

3. If $\underline{x}$ and $\underline{y}$ are linearly independent, then $\underline{x}+\alpha\underline{y}$ and $\underline{x}+\beta\underline{y}$ are linearly dependent if

a) $\alpha \neq \beta$ b) $\alpha = \beta$ c) $\alpha > \beta$ d) $\alpha < \beta$

a) 2 b) 6 c) 3 d) 4

5. Let A= $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, a, b≠0 and r be the rank of A then

a) r = 1 if a = b b) r = 2 for all a,b c) r = 2 if a = b d) r = 2 if a = -b

6. Let V={(x,x,x)|x$\in$R) be a vector space then dimension of V is

a) 1 b) 2 c) 3 d) ∞

7. Let A and B be two square matrices of order n. (A+B)(A-B) = $A^2 - B^2$ if and only if

a) A and B commute b) A and B anticommute

c) A = B d) A = B'

8. If two rows of a square matrix are identical, then its determinant

a) cannot be determined b) is product of diagonal element

c) is equal to zero d) is equal to one

9. Let T : $R^2 \rightarrow R^2$ be the linear transformation defined by T(x,y) = (x,x+y).Then matrix of T with respect to the standard basis is

a) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

10. Let A and B be non-empty subset of a vector space V. Suppose that A$\subseteq$B then

a) If B is linearly independent then so is A.

b) If A is linearly independent then so is B.

c) If B is linearly dependent then so is A.

d) If A is linearly dependent then so is B.

13. If A is m×n matrix over R which of the following is correct

a) row rank(A) = column rank(A) b) row rank(A) > column rank(A)

c) row rank(A) < column rank(A) d) none of these

C) State TRUE or FALSE

1. If A is symmetric matrix then A^n is symmetric matrix.
2. If A and B are two square matrices of order n then rank(AB) = rank(A) + rank(B)
3. Inverse of an orthogonal matrix is transpose of matrix
5. The rank of an singular matrix of order m is m.
7. If A is orthogonal matrix then –A is orthogonal matrix.

8. Every finite dimensional vector space has an orthonormal basis.

D) Problems:

1. Define transpose of a matrix and prove that (A+B)'= A'+ B'
2. Define transpose of a matrix and prove that (AB)'= B'A'
3. Define trace of a matrix and prove that
   i) tr(A+B) =tr(A) +tr(B)
   ii) tr(AB)=tr(BA)
   iii) tr(kA) = ktr(A) ; where k is a constant.
4. If A and B are symmetric matrices, show that AB is symmetric if and only if A and B commute.
5. Show that A^2 is a symmetric matrix, if A is either a symmetric or a skew-symmetric matrix.
6. If A is square matrix then show that
   i) A+A' is symmetric matrix
   ii) A-A' is a skew symmetric matrix.
7. If A and B are square and orthogonal matrices ,then AB and BA are orthogonal matrices.
8. Show that $(AB)^{-1} = B^{-1} A^{-1}$.
9. Define orthogonal matrix and show that for any orthogonal matrix P we have
10. PP'=P'P=I
11. 16 .If A is real symmetric matrix such that A^2 +I =0, show that A is orthogonal.
12. Show that trace(C'AC) =trace(A),if c is orthogonal matrix.
13. Prove that
    a. If A and B are symmetric then $\left[(AB)^1\right]^{-1} = A^{-1}$
    b. C = $X(X'X)^{-1}$ is symmetric and idempotent
    c. The transpose and inverse of an orthogon matrix are equal
    d. All powers of a symmetric arthogon matrix are the matrix itself or an identity matrix.
14. Every non-singular idempotent matrix is an identity matrix.
15. For an orthogonal matrix A we have $A^1 = A^{-1}$

16. For X – symmetric and idempotent and TX symmetric prove that T_X = XTX
17. Define vector space and subspace of vector and determine the following W={(x,y,z)/x+y+z=1} subspace of V=R^3

18. Define linear dependent and independent of vector $\vec{u}_1$=(1,2,-3), $\vec{u}_2$=(1,-3,2), $\vec{u}_3$=(2,-1,5) be vector R^3. Then show that the set B={$\vec{u}_1, \vec{u}_2, \vec{u}_3$) is a basis of $R^{3.}$
19. Define linear combination of vector and let $\vec{u}_1$=(1,1,1), $\vec{u}_2$=(1,1,0), $\vec{u}_3$=(1,0,0) be three vectors in R^n.Show that (3,2,1) is linear combination of above vector.
20. Explain Gram-Schmidt orthogonalization process.
21. Define the partition matrix. Explain the procedure of how to obtain the inverse of general 3×3 matrix by partioning the matrix.
22. Define the rank of matrix and give the illustraton.
23. Explain full rank of factorization method with suitable example.
24. If A and B are the two matrices such that the product is defined then rank(AB) ≤ min(rank(A),rank(B)).
25. If A is a symmetric matrix, then A^c is also symmetric matrix .
26. Prove or disprove: Subset of linearly dependent set of vectors is linearly dependent.
27. Obtain the kroneckar product of two matrices A=$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B=$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.
28. Write a short note on Inverse of a matrix by partition.
29. Using Gram-Schmidt orthogonalization process construct an orthonormal basis for the vector space spanned by $\underline{a1}$ and $\underline{a2}$ as given below

    $\underline{a1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{a2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$
30. Define Kroneckar product of the matrices A and B, where A and B are compatible matrices. State any two properties of Kroneckar product.

Unit 2

A) Define the following terms

1. Determinant of matrix.
2. Inverse of matrix
3. Null space
4. Nullity
5. Permutation matrix
6. Reducible /irreducible matrix
7. Primitive/Imprimtive matrix
8. 12. Idempotent Matrix
9. Nilpotent Matrix
10. Homogeneous system of linear equation.
11. Non-homogeneous system of linear equation.
12. Genaralized inverse of matrix
13. Moore-Penrose g-inverse

B) Choose the correct alternatives of the following:

1. Let A be a n×n non singular real matrix , $n \geq 3$.Then the determinant of the adjoint matrix of A is

   a) Det(A) b) $(\det(A))^{n-1}$ c) $(\det(A))^{n-2}$ d) $(\det(A))^{n}$

2. Let A be an idempotent matrix,then

   a) A=A' b) $A=A^{-1}$ c) $A= A^2$ d) none of these

C) State TRUE or FALSE

1. If A is non- singular matrix then the system AX=b has only one solution.
2. Every matrix has a unique g-inverse.
3. A generalized inverse of a matrix is always exist.
4. Inverse of an square matrix exist if and only if matrix is non -singular.
5. 6. For an idempotent matrix A, $|A|$ = 0 or 1.

D) Problems:

1. Show that Moore Penrose generalized inverse is unique.
2. Define G-inverse of matrix and find G-inverse of following matrix

$$A=\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 2 & 2 \\ -1 & 4 & 5 & 3 \end{bmatrix}$$

3. Define two definition of G-inverse. State equivalence of two definitions.
4. Let G is generalized inverse of a matrix A then ,AG is idempotent is idempotent matrix.

5 . Obtain G-inverse of matrix A. Also verify AGA=A

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 2 \\ 2 & 0 & 4 \end{bmatrix}$$

6. Obtain G-inverse of matrix A.Also verify AGA=A

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \end{bmatrix}$$

7 .Obtain MPG-inverse of matrix A.

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$$

8. Investigate the value of a and b, the following system of equation ha

i) no solution ii) Exactly one solution iii) Infinitely many solutions

$$X+Y+Z = 6$$

$$X+2Y+3Z=10$$

$$X+2Y+aZ=b$$

9. Define trivial and non -trivial solution of the system AX=0.Give an example of each.
10. Find the value of δ so that the following system of equation admits unique solution.

$2X_1 - X_2 + 5X_3 = 4$

$4X_1 + 6X_3 = 1$

$-2X_2 - 4X_3 = \delta$

11. If A is idempotent and A+B = I, then B is idempotent.
12. If A is idempotent then prove that $|A|$=0 or 1.
13. If A and B are two idempotent matrix of order n then (A-B) is idempotent.
14. Define inverse of matrix and explain procedure of adjoint method of inverse

$$A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 10\end{bmatrix}, \text{ Find } A^{-1}$$

15. Prove that the inverse of matrix A is unique.
16. Prove that $(AB)^{-1} = B^{-1} A^{-1}$
17. Define inverse of matrix and prove that reversal rule of inverse for matrices n non-singular square.
18. If I and J matrix I is identity matrix and J is matrix having all elements are equal

$$I_n = \begin{bmatrix}1 & 0 & 0 & ...0\\ 0 & 1 & 0 & ...0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & ...1\end{bmatrix}_{n\times n} \qquad J_n = \begin{bmatrix}1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\\ \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & \cdots & 1\end{bmatrix}$$

Prove that

$$\left(aI_n + bI_n\right)^{-1} = \frac{1}{a}\left(I_n - \frac{b}{a+nb}J_n\right) \text{ for all } a \neq 0$$

19. The value of determinant of matrix is unchanged if the multiple of column is added to another column of matrix.
20. Define Idempotent matrix Prove that (I-A) is idempotent but (A-I) is not idempotent matrix.
21. If A and B are idempotent matrices, then AB is idempotent if A and B commute.
22. Every non- singular idempotent matrix is an identity matrix
23. Define orthogonal matrix and prove that $|A|$=± 1
24. Write a note on n-order determinant.
25. Prove that properties of determinant

i) $|AB| = |A|.|B|$

ii) $|A^{-1}| = |A|^{-1}$

iii) $|A^k| = |A|^k$

26. Write a note on diagonal expansion of matrix and prove that

$$A=\begin{bmatrix} 0 & -a & b & -c \\ a & 0 & -d & e \\ -b & d & 0 & -f \\ c & -e & f & 0 \end{bmatrix}$$

a) $|I+A| = 1+(a^2+b^2+c^2+d^2+e^2+f^2) + |A|$

b) Calculate $|A|$

27. If matrix A have $(a_{ij}) = a$ if $i = 1,2,3,,,,$

$= b$ if $i \neq j=1,2,3,\ldots.$

then show that $\det(A) = [a+(n-1)b](a-b)^{n-1}$

28. If $J_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times n}$ then prove that

i) $J_n^2 = n J_n$

ii) If $\overline{J_n} = \frac{1}{n} J_n$ then prove that $\overline{J_n}$ is idempotent

29. Solve the system

X+Y+Z = 6

X+2Y+3Z=10

X+2Y+4Z=12

30. Find value of a and b for which following system of equation has

(1) No solution

(2) Exactly one solution

(3) Infinitely many solutions

-2y+bz = 3

ax+2z = 2

5x+2y = 1

31. Show that a system of linear equations AX=b is consistent iff rank(A/b) = rank(A).
32. State the properties of generalized inverse.

Unit 3:

A) Define the following terms.

1. Eigen values
2. Eigen vectors
3. Characteristic equation
4. Characteristic polynomial
5. Eigen space of matrix.
6. Geometric multiplicity
7. Algebraic multiplicity

B) Choose the correct alternatives of the following:

1. The characteristic root of the real symmetric orthogonal matrix are

   a) 0 and 1 b) -1 and 1 c) -1 and 0 d) none of these

2. If all the characteristic rots of a matrix A are either 0 or 1 then matrix is

   a) An orthogonal matrix b) an identity matrix.

   c) An idempotent matrix d) none of these

3. If the characteristics root of a matrix A are 4,2,and 1 then

   a) $|A|$ = 8 and tr(A) =7 b) $|A|$ = 8 and tr(A) = 8

   c) $|A|$ = 7 and tr(A) = 8 c) $|A|$ = 7 and tr(A) = 9

4. The characteristic root of idempotent matrix are

   a) $<$ 1 b) ±1 c) 0 or 1 d) >1

5. If eigen values of 2×2 matrix 'A' are 3 and 4 then

   a) det(A)= 12 b) Trace(A) = 7 c) both a) and b) d) det(A)=7

6. Sum of eigen values of a matrix A is equal to

   a) Product of diagonal elements b) sum of diagonal elements

   c) determinant of A d) a positive number always

7. If matrix A has characteristic polynomial f(x) then transpose of a matrix a has characterstic polynomial ………

   a) –f(x) b) f(x) c) f(-x) d) 1/f(x)

C) Theorem1. If A is an n×n matrix and λ is real number, then λ is eigen value of A if and only if

$\det(\lambda I-A)=0$.

Theorem2. Let A be an n×n matrix and λ be an eigen value of A, then eigen space $E(\lambda)$ is subspace of R^n.

Theorem3. If K is positive integer, λ is an eigen value of a matrix A,X is corresponding eigen vector ,then λ^k is an eigenvalue of A^k and X is corresponding eigenvector.

Theorem4. Every n× n matrix satisfies its own characteristics equation.

Theorem5. If A is square matrix, then A and A^t have the same characteristic polynomial.

Theorem6. If S is a real skew symmetric matrix then I-S is non -singular and the matrix $A= (I+S)(I-S)^{-1}$ is orthogonal.

Theorem7. If A is an n×n matrix, then the following are equivalent

a) A is diagonalizable b) A has linearly independent eigenvectors.

Theorem8. Let A be a diagonalizable matrix and let p be the invertible matrix that diagonalizes a then $A^k = PD^kP^{-1}$ where $P^{-1}AP = D$ is a diagonal matrix.

Theorem9. Characteristic roots of real symmetric matrix are real.

Problems:

1. Find eigenvalues of following matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

2. Find all eigenvalues of following matrix.

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$ Also find the eigen space of A corresponding to the smallest eigen value of A.

3. If λ is an eigen value of matrix A, then find eigen value of matrix adjoint of A.
4. Prove that if λ is an eigen value of a square matrix A then λ^m is an eigen value of A^m for every positive integer m.
5. Prove or disprove:
   a) If A is characteristic root of matrix A then (c+λ) is characteristic root of matrix (A+CI).
   b) If λ is characteristic root of A matrix A then $(1+\lambda)^{-1}$ is an characteristic root of $(I+A)^{-1}$.
   c) If is eigen value of matrix A then (t+A) is eigen value of
6. Write a procedure to find generalized inverse for symmetric matrix.
7. Define solution of system of linear equation
8. If A is non -singular matrix then show that its characteristic root is non- zero.
9. Find the characteristic roots of the following matrix

$$A=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

10. Explain a) characteristic roots and characteristic vector of a matrix
    b) Right and left characteristic vectors.

Unit 4: Quadratic Forms

A) Define the following terms:

1. Quadratic forms of n variables.
2. Diagonal Form
3. Canonical Form
4. Positive definite matrix
5. Negative definite matrix
6. Positive semi definite quadratic form
7. Negative semi definite matrix.
8. Classification of quadratic form.
9. Spectral decomposition of real symmetric matrix.

B) Choose the correct alternative of the following :

1. The quadratic form $(X_1+X_2)^2$ is

   a)Positive definite b) negative definite

   c)positive semi definite d) negative semi definite

2. which of the following quadratic form is not positive definite?

   a)$X_1^2 + X_2^2$ b) $X_1^2 + X_2^2 + X_1X_2$ c) $X_1^2 + X_2^2 - \frac{1}{2} X_1X_2$ d) $X_1^2 - X_2^2$

3. The quadratic form $X_1^2 + X_2^2 + X_3^2$ is

   a)Positive definite b) negative definite

   c)positive semi definite d) negative semi definite

4. The quadratic form $X^2 - 2XY + Y^2$ is

   a)Positive definite b) negative definite

   c)positive semi definite d) negative semi definite

C) State TRUE or FALSE :

1. The symmetric matrix A of the quadratic form $(X_1-X_2)^2$ is, $A=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
2. Let A be an idempotent matrix, then the value of $\underset{X}{Sup} \frac{X'AX}{X'X}$ is one.
3. Algebraic multiplicity is always greater than geometric multiplicity.
4. A quadratic form Q=X'AX is positive definite iff the eigen values of matrix A is positive.

D) Problems:

1. Prove that a quadratic form X'AX is positive definite if and only if the characteristics root of matrix A are all positive.
2. Necessary and sufficient condition for existence of positive definite quadratic form.
3. Write matrices of the given quadratic form of n-array.
4. Write definiteness of given quadratic form.
5. Define a positive semi definite quadratic form. Prove that X'AX is positive semi definite quadratic form under certain conditions to be stated.

6. Show that if A is real symmetric matrix then there exist a real orthogonal matrix P such that P'AP= diag ($\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$) where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are characteristic roots of A.
7. Prove or disprove

   For a symmetric matrix A

   i) The quadratic form X'AX is positive semi definte if A is idempotent

   ii) The quadratic form X'AX is positive definite if A is orthogonal.

   iii) The quadratic form $X'A^2X$ is always positive semi definite.
8. Define quadratic form show that quadratic form is invariant under non -singular transformation.
9. Reduce the quadratic form $X_1^2 + 2X_2^2 + 3X_3^2 + 2X_1X_2 + 2X_2X_3 - 2X_3X_1$ to a canonical form
10. Let $A_{n\times n}$ be a symmetric matrix with characteristic roots $\lambda_1 \geq \lambda_2 \geq, \ldots\ldots, \geq \lambda_n$ prove that $\underset{X}{Sup} \dfrac{X'AX}{X'X} = \lambda_1$
11. Show that a quadratic form can be transformed to a diagonal form containing only square terms.
12. If X'AX is real quadratic form with rank(A)= r , show that there exist an orthogonal transformation X=PY such that X'AX is transformed to $\sum_j \lambda_i Y_j^2$ where λ_j ; j= 1,2 ,….,r are the non- zero characteristic roots of A.
13. Examine the nature of the following quadratic form xy + yz +xz.
14. Show that a non- singular symmetric matrix A is congruent to its inverse.
15. Show that areal symmetric matrix A is positive definite if and only if there exists a non-singular matrix Q such that A=Q'A
16. Find matrix P that diagonalizable A=$\begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$ and determine P^{-1} AP.
17. Find the maximum and minimum values of the quadratic form subject to the constraint $X_1^2 + X_2^2 + X_3^2 = 1$ and determine the values of X_1 , X_2 ,X_3 at which maximum and minimum occur where $Q = 2X_1^2 + 2X_2^2 + 2X_3^2 + 2X_1X_2$.

18. Reduce the quadratic form Q $=6X_1^2 +35X_2^2 +11X_3^2 +34X_2X_3$.

19. Examine for definiteness of the following quadratic form

a) Q $=9X_1^2 +4X_2^2 +4X_3^2 +8X_2X_3+12X1X3+12X_1X_2$.

b) Q $=X_1^2 -2X_1X_2+ X_2^2-X_3^2$.

20. Describe the classification of quadratic form.

21. Explain the spectral decomposition of a symmetric matrix. Obtain for the same

$$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

22. Explain the spectral decomposition of a symmetric matrix. Obtain for the same

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$ Hence obtain A^2.