Anekant Education Society's

TuljaramChaturchand College, Baramati

Department of Mathematics

Class: M.Sc –I

Question Bank

Title of Paper : Real Analysis **Sub Code:**MAT4101

Short Answer Type Question.

1) Write Short Note on

a) Metric Space

b) Normed Linear Space

c) Inner Product Space.

2)Verify that $\|.\|_1$, $\|.\|_2$,$\|.\|_\infty$ define norms on R^2.

3) Give an example of a Metric that do not come from a Norm.

4) Show that C^n with $\langle z, w\rangle = z_1w_1 + z_2w_2 + \ldots + z_nw_n$. is an inner product space.

5) Prove that Inner product always give rise to a Norm.

6) State & Prove Parallelogram Equality.

7) Write Short Note on $\|.\|_1$ &$\|.\|_\infty$.

8) Prove that $\| x \| = 1/3\| x \|_1 + 2/3\| x \|_\infty$defines a Norm on$R^n$.

9) Show that if E is Compact Subset of a Metric space, then E is closed.

10) Prove that every convergent sequence is Cauchy.

11) Is it true that $int(A\cup B) = int(A) \cup int(B)$? Prove or give a counter example what if union is replaced by intersection.

12) Prove that a totally bounded set is necessarily bounded.

13) If f is measurable then prove that $|f|$ is measurable.

14) For $1\leq p< \infty$ define $L^p(\mu)$ & prove that it is a Linear space.

15) Prove that in an inner product space $(V, < .,.>)$, f and g are orthogonal implies that $\|f\|^2 + \|g^2\| = \|f+g\|^2$.

16) Give an example to show that pointwise convergence does not imply uniform convergence.

17) Define measurable sets in R^n& show that family of measurable sets is σ - ring.

18) Show that Riemann integrable function is also Lebesgueintegrable.

19) If f & g are measurable functions then show that fg is measurable.

20) If f & g are measurable functions, Then so are f + g, fg, f_+& f_-

21) Is the Cantor set compact? Explain.

22) What is the interior of the Cantor set?Explain.

23) Write short note on Lebesgue measure.

24) Prove that outer measure of R^d is infinite.

25) Prove that outer measure of Cantor set is zero.

26) Write short note on Measurable functions.

27) Show that the Lebesgue integral is Monotone i.e. $\int f dm \leq \int g dm$ on E

Whenever $f(x) \leq g(x)$ $\forall x \in E$.

28) Write short note on

i) Measure space.

ii)$L^p(\mu)$

iii) $L\infty(X, \mu)$

29) Show thatFor $1 \leq p \leq \infty$, $L^p(\mu)$ is a linear space.

30) Prove that $L^p(\mu)$ is a Normed linear space, with norm given by

$\| f \|_p = (\int |f|_p d\mu)^{1/p}$.

ANSwer IN ONE SENTENCE

1) Define the following terms.

i)a Measure.

ii) Separable metric space

iii)complete metric space

iv) Sup Norm on C (a, b).

v) Open set

vi) Sequentially Compact set.

vii) Dense set

viii) Compact set

ix) Banach space

x) Hilbert space

xi) Outer Measure.

xii) Lower Riemann Sum

xiii) Upper Riemann Sum

xiv) Riemann integrable function.

xv) Borel Set.

xvi) Counting measure.

2) Let M be any set with discrete metric. What are the open sets?

3) Give an example of closed & bounded set in a metric space that is not compact.

4) Give an exampleto show that arbitrary intersection of open set in a metric space is not open

5) State the following:

a) Heine Borel Theorem.

b) Lebesgue Monotone Convergence theorem.

c) Holder’s Inequality.

d) Fatou’s Lemma.

e) Cauchy BunyakowskiiShwarz Inequality.

f) Bessel’s Inequality.

g)Parseval’s theorem

h) Baire Category Theorem.

i) Weierstrass approximation theorem.

6) Define a simple function

7) Define aEquicontinuity.

TRUE FALSE WITH JUSTIFICATION

1)The outer measure of a point is 1.

2) The outer measure of a closed cube is its volume.

3) The outer measure of R^d is finite.

4) Every open set in R^d is measurable.

5) A countable union of measurable set is measurable.

6) Closed sets are non-measurable.

7) The Lebesgue measure in R^d is translation in variant.

8) Every Continuous function need not be measurable.

9) C ([0,1]) is finite dimensional.

10) The step Functions are not dense in $L^p(\mu)$.

LONG ANSWER TYPE QUESTION

1) State &prove CauchyBunyakowskiiShwarz Inequality.

2) Prove that a subset of a metri space is compact iff it is sequentially compact.

3) State &prove HeineBorel Theorem.

4) State &Prove Ascoli-ArzelaTheore.

5) Show that C([a, b]; R) with the supremum norm is Complete.

6) Show that if M is metric space, A $\subseteq M$ is compact & U$\subseteq M$ is open then A\U is compact.

7) Consider metric spaces M_1& M_2.

a) Prove that a function f : $M_1 \rightarrow M_2$ is continuous if & only if the inverse image of anopen set in M_2 is open in M_1.

8) Is ℓ^1 separable? Prove your assertion.

9) Is ℓ^∞ separable? Prove your assertion.

10) Prove that in a metric space every Cauchy Sequence is bounded.

11) Define $\| f \|_1 = \int_a^b |f(x)| dx$ on C([a, b]). Show that this does, indeed, define a norm. Does this norm come from an inner product.

12) Show that C([a, b]; $\| . \|_2$) is not Complete.

13) Prove that $\ell^1$is complete.

14) On R^n , show that the three norms $\| . \|_1$,$\| . \|_2$,& $\| . \|_\infty$ are equivalent.

15)Prove that If A $\in \mathcal{E}$&$\epsilon > 0$then there exists a closed set F $\in \mathcal{E}$ Ɛ an open set G $\in \mathcal{E}$ such that F$\subseteq A \subseteq$ G &m(F) $\geq m(A) - \varepsilon$&m(F) $\leq m(A) + \varepsilon$

16) Show that m is a measure on $\mathcal{E}$.

17) State & prove Countable Subadditivity of outer measure.

- Let $\mathcal{M}_F$ denote the collection of subsets A of R^n such that D(A_k, A) $\rightarrow$ 0 as k $\rightarrow \infty$ for some sequence of sets $A_k \in \mathcal{E}$
- $\mathcal{M}$denote the collection of subsets of R^n that can be written as countable union of sets in $\mathcal{M}_F$.

18) Prove that m* is additive on $\mathcal{M}_F$.

19) Let A $\in \mathcal{M}$ then showthat A$\in \mathcal{M}_F$ iffm*$< \infty\infty$

20)) Prove that $\mathcal{M}$ is a $\sigma - ring$ m* is countably additive on $\mathcal{M}$.

21) Prove that if f is a real valued functions defined on R^n then ∃ a sequence

$\{ s_k\}$ of simple functions such that $\lim s_k(x) = f(x)$, $\forall\ x \in R^n$.

22)Assume that $f \geq 0$ is measurable & that $A_1, A_2, \ldots \in \mathcal{M}$ are pairwise disjoint.Then ,$\int_{\cup_{k=1}^{\infty} Ak} f dm = \sum_{k=1}^{\infty}(\int_{Ak} f dm)$

23)) State & prove Lebesgue Monotone Convergence theorem.

24) State &prove Fatou's Lemma.

25) State &prove Lebesgue Dominated Convergence theorem.

26) Prove that if f is Riemann integrable on [a, b] then f is Lebesgueintegrable on [a, b]. &$\int f dm = \int_a^b f(x)dx$.

27)) Show that For $1 \leq p \leq \infty$, $L^p(\mu)$ is complete.

28) Prove that a normed linear space (X, ‖.‖) is complete iff$\sum_{i=1}^{\infty} f$i converges whenever $\sum_{i=1}^{\infty} \|f i\|$ converges.

29) Prove that the step functions are dense in$L^p(\mu)$, for each $1 \leq p \leq \infty$

30) Assume that μ is nonnegative, additive function defined on a ring R, prove that μ is monotone.

31) Prove that every continuous function is measurable.

32) Prove that f & g are equal almost everywhere iff$\int f dm = \int g dm$ on E∀ measurable set E.

33) Prove that f = 0 a.e. if $\int_E f dm = 0$ for every $E \in \mathcal{M}$.

34) Which Characteristic functions are Lebesgueintegrable on R? Is the Characteristic functions of the rational numbers integrable on the unit interval? If so what is the value of this integral.

35)Show that for each $f \in L\infty$ L' $|f(x)| \leq \| f \|\infty$.

36) State & Prove Parseval's theorem.

37) State & Prove Baire Category Theorem.

38) State & Prove Weierstrass approximation theorem.