Anekant Eduction Society's

Tuljaram Chaturchand College,Baramati

Department of Mathematics

Question Bank

Class:F.Y.B.Sc.

Title of Paper:Calculus-I

Subject Code:MAT1102

Short Answer Questions

1. Prove that $\sqrt{2}$ is not rational number.
2. Prove that $\sqrt{2}+\sqrt{3}$ is not rational number.
3. Define the term: Algebraic numbers.
4. Prove that any rational number is algebraic.
5. Prove that $\sqrt{3}$, $\sqrt[3]{5}$ are algebraic numbers.
6. Find Supremum and Infimum of the following set if exists, $\cup_{n=1}^{\infty}[2n, 2n+1]$
7. Find Supremum and Infimum of the following set if exists, $\{1, \pi, e, -7, -2\}$
8. Find Supremum and Infimum of the following set if exists, $\mathbb{N}$
9. Find Supremum and Infimum of the following set if exists, $\mathbb{R}$
10. Define the term Limit of Sequence.
11. Using definition of limit of a sequence, prove that
12. Using definition of limit of a sequence, prove that$\lim_{n\to\infty} \frac{3+\frac{1}{n}}{7-\frac{1}{n}} = \frac{3}{7}$
13. Using definition of limit of a sequence, prove that $\lim_{n\to\infty} \frac{4n^3+3n}{n^3-6} = 4$
14. State and prove Archimedian property.
15. Find greatest lower bound and least upper bound of the following set if exists, $\{1, \pi, e, -7, -2, 100, 10001\}$
16. Find greatest lower bound and least upper bound of $\mathbb{N}$ if exists. $\mathbb{N}$.
17. Define the term : Convergence Sequence.
18. Check convergence of seuence,
19. Write out the first five terms of the following sequence. $s_n = \frac{1}{3n+1}$
20. Write out the first five terms of the following sequences. $bn = \frac{3n+1}{4n-1}$

21. Determine whether the sequence converges and, if it converges, give its limit.
$a_n = \frac{n}{n+1}$.

22. Determine whether the sequence converges and, if it converges, give its limit.
$a_n = \frac{n^2+3}{n^2-3}$.

23. Determine whether the sequence converges and, if it converges, give its limit.
$sin(n\pi)$.

24. Determine whether the sequence converges and, if it converges, give its limit.
$(-1)^n$.

25. Give examples of a sequence (x_n) of irrational numbers having a limit $\lim x_n$ that is a rational number.

26. Give examples of A sequence (r_n) of rational numbers having a limit $\lim r_n$ that is an irrational number.

27. Using definition of limit of sequence, prove that $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

28. Show that the sequence $a_n = (-1)^n$ does not converge.

29. Let (s_n) be a convergent sequence of real numbers such that $s_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = s \neq 0$. Prove $inf\{|s_n|/ : n \in \mathbb{N}\} > 0$.

30. Determine the limits of the following sequences, and then prove your claims.
(a) $a_n = \frac{n}{n^2+1}$
(b) $b_n = \frac{7n-19}{3n+7}$

31. Let (s_n) be a sequence of nonnegative real numbers, and suppose $lim_{n\to\infty} s_n = 0$. Prove $\lim_{n\to\infty} s_n = 0$.

32. Let (t_n) be a bounded sequence, i.e., there exists M such that $|t_n| \leq M$ for all n, and let (s_n) be a sequence such that $\lim_{n\to\infty} s - n = 0$. Prove $lim(s_n t_n) = 0$.

33. Show the following sequences do not converge.
(a) $cos(\frac{n\pi}{3})$
(b) $s_n = (-1)^n n$

34. Prove that, $\lim_{n\to\infty}(\frac{1}{n^p}) = 0$ for $p > 0$.

35. Prove that, $lim_{n\to\infty} a_n = 0$ if $|a| < 1$.

36. Prove that, $lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

37. Prove that, $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ for $a > 0$.

38. Find $\lim_{n\to\infty} \frac{n^2+3}{n+1}$.

39. Prove that $\lim_{n\to\infty} \frac{n^3+6n^2+1}{4n^3+3n-4} = \frac{1}{4}$.

40. Let $s_1 = 1$ and for $n > 1$. let $s_{n+1} = s_n + 1$.
(a) List the first four terms of (s_n).
(b) It turns out that (s_n) converges. Assume this fact and prove the limit is $\frac{1}{2}(1+\sqrt{5})$.

41. Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n > 1$.
(a) Show if $a = \lim_{n\to\infty} x_n$, then $a = \frac{1}{3}$ or $a = 0$.
(b) Does $\lim_{n\to\infty} x_n$ exist? Explain.

42. Verify $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ for $a \neq 1$.

43. Find $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.

44. Calculate $\lim_{n\to\infty}(1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n})$.

45. Define the terms : a) Monotone sequence b) Cauchy Sequence

46. Give an example of bounded sequence which is not Cauchy.

47. Prove that , every convergent sequence is Cauchy. Is converse true? Justify.

48. Prove that, a sequence is convergent if and only if it is Cauchy.

49. Which of the following sequences are increasing? decreasing? bounded?
(a) $\frac{1}{n}$
(b) $\frac{(-1)^n}{n^2}$
(c) n^5
(d) $sin(\frac{n\pi}{7})$
(e) $(-2)^n$

50. Let (s_n) be a sequence such that, $|s_{n+1} - s_n| < 2^{-n}$ for all $n \in \mathbb{N}$. Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

51. Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove (σ_n) is an increasing sequence.

52. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n > 1$.
(a) Find s_2, s_3 and s_4
(b) Use induction to show $s_n > \frac{1}{2}$ for all n.
(c) Show (s_n) is a decreasing sequence.

53. Define the term : Subsequence

54. State Bolzano-Weierstrass Theorem.

55. Define the term

Sub-sequential Limit.

56. Consider the sequences defined as follows: $a_n = (-1)^n, b_n = \frac{1}{n}, c_n = n^2, d_n = \frac{6n+4}{7n-3}$

(a) For each sequence, give an example of a monotone subsequence.
(b) For each sequence, give its set of subsequential limits.
(c) For each sequence, give its lim sup and lim inf.
(d) Which of the sequences converges? diverges to ∞? diverges to $-\infty$?
(e) Which of the sequences is bounded?

57. If $\lim_{n\to\infty} |\frac{s_{n+1}}{s_n}|$ exists and equals L, then prove that $\lim_{n\to\infty} |s_n|^{\frac{1}{n}}$ exists and equals L.

58. Define the terms : a) Series b) Convergence of series

59. State Ratio Test for series.

60. State Comparison Test series.

61. If the series $\sum a_n$ converges then prove that $\lim_{n\to\infty} a_n = 0$.Is converse true? Justify.

62. State Root Test for series.

63. Determine which of the following series converge. Justify your answers.
   (a) $\sum \frac{n^4}{2^n}$
   (b) $\sum \frac{2^n}{n!}$
   (c) $\sum (-1)^n$
   (d) $\sum \frac{n^2}{n!}$

64. Prove that if$\sum |a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_n b_n$converges.

65. Calculate a) $\sum_{n=1}^{\infty} (\frac{2}{3})^n$ b)$\sum_{n=1}^{\infty} (\frac{n}{2^n})$

66. Prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

67. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

68. Suppose $\sum a_n = A$ and $\sum b_n = B$ where A and B are real numbers. Use limit theorems , to prove the following.
   (a) $\sum a_n + b_n = A + B$
   b)$\sum ka_n = kA$ for $k \in \mathbb{R}$

69. Check the convergence of series $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

70. Define the geometric series and discuss the convergence.

71. Define the terms : a) Absolute Convergent b) Conditionally Convergent

72. State and Prove rational zero theorem for polynomials.

73. Consider the polynomial equation $x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$, where the coefficients$c_0, c_1, ..., c_{n-1}$ are integers and $c_0 \neq 0$. Any rational solution of this equation must be an integer that divides c_0.

74. Prove that $\sqrt{2 + \sqrt[3]{5}}$ is not a rational number.

Long Answer Questions:

1. State and prove Completeness Axiom.

2. If S is any nonempty subset of $\mathbb{R}$ and bounded below then , prove that S has infimum.

3. State and prove Density theorem for $\mathbb{Q}$.

4. Let S be any bounded subset of $\mathbb{R}$. Prove that $infS \leq supS$.

5. Let S and T are any bounded subset of $\mathbb{R}$ and $S \subset T$. Prove that $supS \leq supT$.

6. Let S and T are any bounded subset of $\mathbb{R}$ and $S \subset T$. Prove that $infT \leq infS$.

7. If $a > 0$ and $b > 0$ are any real numbers ,then prove that there exists a positiv integer such that $na > b$.
8. Let S and T are any bounded subset of $\mathbb{R}$ and $S + T = \{s + t/s \in S, t \in T\}$ Prove that $Sup(S + T) \leq SupS + SupT$.
9. Let S and T are any bounded subset of $\mathbb{R}$ and $S + T = \{s + t/s \in S, t \in T\}$ Prove that $Inf(S + T) \geq InfS + InfT$.
10. Prove that , every convergent sequence is bounded. Is Converse true?
11. Give an example of a sequence which is bounded but not convergent.
12. Prove that, if the sequence (s_n) converges to s and k is in $\mathbb{R}$, then the sequence (ks_n) converges to ks.
13. Prove that, if (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$.
14. Prove that ,if (s_n) converges to s and (t_n) converges to t, then $(s_n t_n)$ converges to st.
15. Prove that,if (s_n) converges to s, if $s_n \neq 0$ for all n, and if $s \neq 0$, then $\frac{1}{s_n}$ converges to $\frac{1}{s}$.
16. Let (s_n) be any sequence in $\mathbb{R}$, and let S denote the set of sub-sequential limits of (s_n).Then prove that
(i) S is nonempty.
(ii) sup S = lim sup s_n and inf S = lim inf s_n
(iii) $\lim s_n$ exists if and only if S has exactly one element.
17. Let (s_n) be any sequence. Then prove that There exists a monotonic subsequence whose limit is $\lim sups_n$.
18. Let (s_n) be any sequence. Then prove that there exists a monotonic subsequence whose limit is $\lim inf s_n$.
19. Prove that, if (s_n) converges to a positive real number s and (t_n) is any sequence, then $\lim sups_n t_n = s\ limsup(t_n)$. Here we allow the conventions $s(+\infty) = +\infty$ and $s(-\infty) = -\infty$ for $s > 0$.
20. Prove that ,if the sequence (s_n) converges then every subsequence converges to the same limit.
21. Prove that ,Every sequence (s_n) has a monotonic subsequence.
22. Prove that, every bounded sequence has a convergent subsequence.
23. Prove that, every monotone bounded sequence is convergent.
24. Prove that, if (s_n) is an unbounded increasing sequence, then $\lim_{n\to\infty} s_n = +\infty$
25. Prove that, if (s_n) is an unbounded decreasing sequence, then $\lim_{n\to\infty} s_n = -\infty$.
26. Prove that, every Cauchy sequence is bounded.Is converse true? Justify.